Physics Factsheet

September 2001 **Number 20**

Simple Harmonic Motion – Basic Concepts

This Factsheet will:
- **explain what is meant by simple harmonic motion**
- **explain how to use the equations for simple harmonic motion**
- **describe the energy transfers in simple harmonic motion**
- **show how to represent simple harmonic motion graphically**
- **explain the link between simple harmonic motion and circular motion**

Later Factsheets will deal with specific examples of simple harmonic motion, such as the simple pendulum and mass-spring system, and damped oscillatory motion.

What is simple harmonic motion?
Simple harmonic motion (SHM) is one form of oscillatory motion. SHM occurs when the resultant force acting on a body has particular properties:

> A body performs SHM if it is acted upon by a force
> - the magnitude of which is proportional to the distance of the body from a fixed point
> - the direction of which is always towards that fixed point.

A simple example of SHM can be observed by attaching a mass to a spring, then pulling the mass down and releasing it. It will bob up and down – this motion is SHM.

In this case, the "fixed point" in the above definition is the **equilibrium position** of the mass – where it was before it was pulled down.

The resultant force acting on the mass is composed of its weight and the tension in the spring. This will always be "trying" to pull the mass back to the equilibrium position, and the further away from equilibrium the mass is, the stronger this force will be.

Equations for SHM
The definition of SHM above can be expressed in the form of an equation:

$$F = -kx$$

Note that the minus sign appears because the force is directed back towards the fixed point.

For the sake of convenience, this is more usually written:

$$F = -m\omega^2 x$$

The reason why this form is more convenient will become apparent shortly.

By using "$F = ma$", this leads to the equation

$$a = -\omega^2 x$$

This equation leads to the following equations (but you do not need to know how unless you are studying SHM in A-level Maths!):

$v^2 = \omega^2(r^2 - x^2)$ where v = speed
x = displacement
r = amplitude = maximum displacement

$x = r\cos\omega t$ where t = time

This equation assumes that the particle starts at a point of maximum displacement – in the case of the spring example, this would mean it starts from the "pulled down" position

If we started timing from the equilibrium position then the displacement would be given by $x = r\sin\omega t$

$v = -r\omega\sin\omega t$

Again, this assumes the particle starts at the point of maximum displacement. – if it starts from equilibrium, we'd have $\mathbf{v} = r\omega\cos\omega t$

Some helpful maths

1. $\cos\omega t$ means $\cos(\omega \times t)$ – so you have to work out ωt first, then find its cosine.

2. **Radians** are another way of measuring angles, rather than degrees. Angular velocities (see also Factsheet 19 Circular Motion) are measured in radians per second (rad s^{-1}). **When you are using functions like cosωt, you will need to work in radians.** The best way to do this is to put your calculator into radians mode, enter the value you have and work out its cosine as normal.
   - On a standard scientific calculator, you can change in and out of radians mode using the button labelled DRG (where D = degrees and R = radians). Some graphical calculators work in this way too; on others you need to go through the setup menu.
   - If you have to convert between degrees and radians (which you usually won't):
     - To change degrees to radians, multiply by $\pi/180$.
     - To change radians to degrees, multiply by $180/\pi$

3. sine and cosine repeat themselves every 360° (= 2π radians). This is because 360° is a full circle, so if you add 360° to an angle in degrees, you get back to where you started. We can say that sine and cosine have a **period of repetition** of 360° or 2π radians

We can deduce some more results from these equations:

Maximum speed
If we look at the speed equation $v^2 = \omega^2(r^2 - x^2)$, we can see that the bigger x^2 is, the smaller v^2 is and vice versa. The smallest possible value of x^2 is 0. Putting $x^2 = 0$ in gives us $v^2 = \omega^2r^2$, hence:

$v_{max} = \omega r$; this occurs when $x = 0$

Maximum acceleration
Since $a = -\omega^2x$, the acceleration will be greatest in magnitude when x is greatest in magnitude, hence

maximum magnitude of $a = \omega^2r$

Acceleration at any time
Using $a = -\omega^2x$ together with $x = r\cos\omega t$, we have

$a = -\omega^2\cos\omega t$

Period and frequency
We know $x = r\cos\omega t$.
Since cosine repeats every 2π radians (see maths box on page 1), the displacement will first return to its intial value when $\omega t = 2\pi$.

The time required for this is the **period** of the motion, T. From above, we have

$$T = \frac{2\pi}{\omega}$$

The **frequency** of the motion is the number of oscillations per second.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

So $\omega = 2\pi f$

It is worth noting that the **period and frequency do not depend on the amplitude** of the motion.

Exam Hint: You may need to know how to derive the expressions for maximum speed and maximum acceleration. Other equations can be learnt – but check your formula sheet, to make sure you do not waste time learning equations you will be given.

The equations are summarised in Table 1 below

Table 1. SHM equations sorted by type

acceleration equations	speed equations	displacement equations	period/frequency equations
$a = -\omega^2x$	$v^2 = \omega^2(r^2 - x^2)$		$T = \frac{2\pi}{\omega}$
$a = -\omega^2\cos\omega t$	$v = -r\omega\sin\omega t$	$x = r\cos\omega t$	$\omega = 2\pi f$
$a_{max} = \omega^2r$	$v_{max} = \omega r$	$x_{max} = r$	

Using the equations
At first sight, there may seem to be a bewildering number of equations to choose from. Here are some strategies to ensure you use the correct one(s):

- Focus on ω. If you are given the period or frequency, use these to find ω before doing anything else. Similarly, if you are asked to find the period or frequency, find ω first.
- Write down what you know and what you want. Then choose the equation with just these symbols in it.
- You won't need to use the equations for the maximum value of displacement, velocity or acceleration unless maximum values are specifically mentioned.
- If time is not mentioned anywhere, you are probably going to be using $v^2 = \omega^2(r^2 - x^2)$.

You may also find the following useful:
- The time required for the body to go between a maximum value of displacement to the equilibrium position is one quarter of the period
- The time required for the body to go between one maximum of displacement to the other (i.e. the two "ends" of the motion) is half of the period

The way to approach problems is best seen from worked examples:

Example 1. An object is oscillating with simple harmonic motion. Its maximum displacement from its equilibrium position is 0.2m. The period of the motion is 0.1 s. Find its speed when it is 0.06m from its equilibrium position

Since we are given T, our first step is to calculate ω.

$$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T} = 62.8 \text{ rads}^{-1}$$

We now know: r (= max displacement) = 0.2m
$\omega = 62.8$ rad s^{-1}
$x = 0.06$ m
We want: $v = ?$

Since r, ω, x and v are involved (and not t), we use
$v^2 = \omega^2(r^2 - x^2)$
$v^2 = 62.8^2(0.2^2 - 0.06^2)$
$v^2 = 143.5$
$v = 12.0$ ms^{-1}

Example 2. A mass is moving with simple harmonic motion; its displacement was at a maximum of 1.1m when t = 0. Its maximum speed is 0.33ms⁻¹. Find:
a) its frequency;
b) its speed after 2.0 seconds.

a) Since we are asked for frequency, we need to find ω first
So we have: $r = 1.1$ m
$v_{max} = 0.33$ ms^{-1}
$\omega = ?$
So we need to use the equation with these three letters in it:
$v_{max} = \omega r$
$0.33 = \omega 1.1$
$\omega = 0.33 \div 1.1 = 0.30$ rads^{-1}

Now we have ω, we find f using
$\omega = 2\pi f$
$0.3 = 2\pi f$
$f = 0.30 \div (2\pi) \approx 4.8 \times 10^{-2}$ Hz

b) We have t = 2.0 s; ω = 0.30 rads⁻¹; r = 1.1 m; v = ?
So we must use:
$v = -r\omega\sin\omega t$
$= -1.1 \times 0.30 \times \sin(0.30 \times 2.0)$
$= -1.1 \times 0.30 \times 0.565$
≈ -0.19 ms^{-1}

Exam Hint:
1. Make sure you do 0.30 × 2.0 before finding the sine
2. Ensure your calculator is in radians mode!

Example 3. A heavy body is performing simple harmonic motion. Its displacement is at its maximum value of 0.40 m when t = 0. It first reaches a point 0.20 m from its equilibrium point after 3.0 s.

a) Find the period of the motion.

b) Explain why the words "first reaches" are important for your calculation in a).

c) Find the body's displacement when its speed is 5.0×10^{-2} ms^{-1}

a) *Since we are asked for the period, we first need to find ω*

We have: $r = 0.40$ m, $x = 0.20$ m, $t = 3.0$ s, $\omega = ?$

So we use

$x = r\cos\omega t$

$0.20 = 0.40\cos(\omega \times 3.0)$

To solve this sort of equation, we must get the part with the cos on its own first:

$0.20 \div 0.40 = \cos(3\omega)$

$0.50 = \cos(3\omega)$

*Now we must use $\cos^{-1}$ (using INV COS on the calculator) to find the angle – **in radians** – whose cos is 0.5:*

$3\omega = \cos^{-1}(0.50) = 1.05$

$\omega = 0.35$ rad s^{-1}

Now we can find $T = \frac{2\pi}{\omega} = 18$ s

b) *Since SHM is repetitive, there will be other times when the body reaches this displacement.*

c) *We have* $\omega = 0.35$ rad s^{-1}, $r = 0.40$ m, $v = 5.0 \times 10^{-2}$ ms^{-1}, $x = ?$

So we use

$v^2 = \omega^2(r^2 - x^2)$

$0.05^2 = 0.35^2(0.4^2 - x^2)$

$0.0025 = 0.122(0.16 - x^2)$

$0.0025 \div 0.122 = 0.16 - x^2$

$0.0205 = 0.16 - x^2$

$x^2 = 0.16 - 0.0205 = 0.1395$

$x = 0.37$ m

Example 4. A body performs simple harmonic motion. Its motion is timed from a point of maximum displacement. Two seconds later, it reaches the equilibrium position for the first time. Its maximum acceleration is 0.60 ms^{-2}. Find:

a) its period;

b) its maximum speed.

a) *Initially, we do not seem to have enough information to use any equation. But we do know that it takes two seconds to move from a point of maximum displacement to the equilibrium position – which corresponds to a quarter of the period.*

So the period is $4 \times 2 = 8$ *seconds*

b) *For any further calculations, we will need* $\omega = \frac{2\pi}{T} = \frac{2\pi}{8}$

$= 0.785$ rad s^{-1}

we know: $a_{max} = 0.60$ ms^{-2}

So use $a_{max} = \omega^2 r$

$0.6 = 0.785^2 r$

$r = 0.6 \div 0.785^2 = 0.973$ m

We need $v_{max} = \omega r = 0.785 \times 0.973 = 0.76$ ms^{-1}

Note: we could have obtained this by dividing a_{max} by ω.

Typical Exam Question

A body performs SHM with a period 3 seconds. Timing starts at one of the extremes of displacement of the body.

Determine the next three times when:

(a) displacement is at an extreme of the motion; [3]

(b) velocity is zero; [3]

(c) acceleration is zero. [3]

(a) it will be at the other extreme half a cycle later, so the times are 1.5 s✓, 3.0 s ✓, 4.5 s ✓

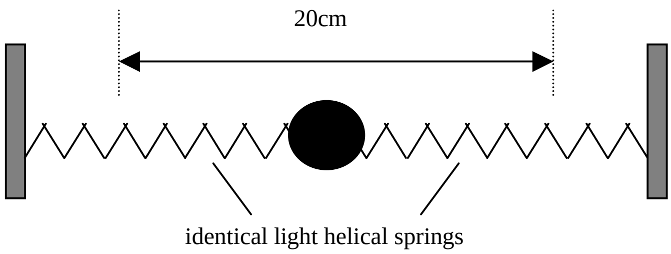
(b) Velocity is zero when $\omega^2(r^2 - x^2) = 0$ *– so when* $x = \pm r$ ✓

So times are 1.5 s, 3.0 s and 4.5 s as in (a). ✓✓

(c) Acceleration = $-\omega^2 x$. *So acceleration is zero when body is at equilibrium position.*✓ *This is midway between the times it is at the extremes. So we have 0.75 s, 2.25 s, 4.75 s* ✓✓

Typical Exam Question

The body in the diagram performs simple harmonic motion between the points shown as dotted lines. The period of the motion is 2s.



Calculate the following quantities:

(a) amplitude; [1]

(b) maximum acceleration; [2]

(c) maximum speed. [2]

(a) amplitude = distance from equilibrium to max displacement pt

The distance shown is twice that – so amplitude = 10cm = 0.10m✓

(b) We need $\omega = \frac{2\pi}{T} = \pi$

$a_{max} = \omega^2 r = \pi^2(0.10)$ ✓ $= 0.99$ ms^{-2} ✓

(c) $v_{max} = \omega r = 0.10\pi$ ✓ $= 0.31$ ms^{-1} ✓

Exam Hint: - Many candidates lose marks through omitting to change their answers to SI units – in the above example, if the amplitude had not been changed into metres, the values for maximum acceleration and speed could have been inconsistent.

Energy Transfers in SHM

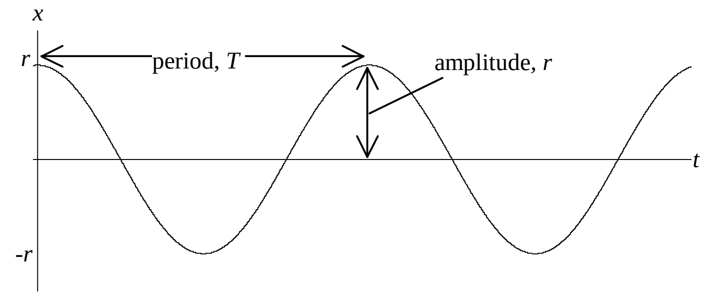
In SHM, with no external forces (so no "damping" or "forcing"), the total energy of the oscillating system remains constant.

Although total energy remains constant, energy is transferred between kinetic energy and potential energy – when the body is at the equilibrium position, it is moving at its fastest, so its kinetic energy is maximum, and when it is at the points of maximum displacement, its speed is zero so its kinetic energy is also zero. These energy transfers are best represented graphically (see next section).

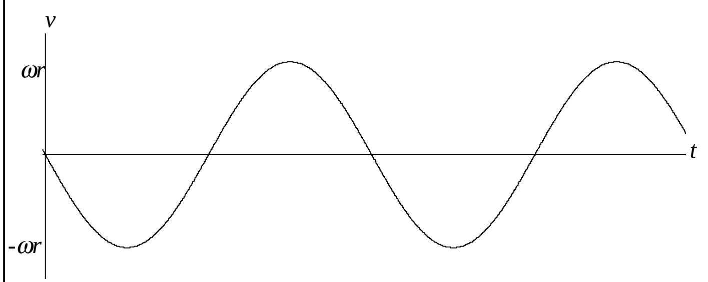
Graphical Representation of SHM

Displacement, speed and acceleration against time
The graphs of these graphs, as might be expected from their respective equations, produce a standard "sine wave" shape:

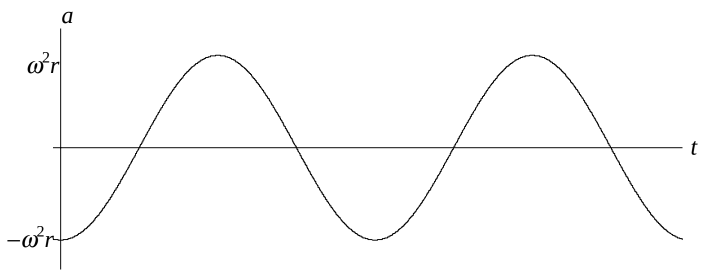
displacement



speed



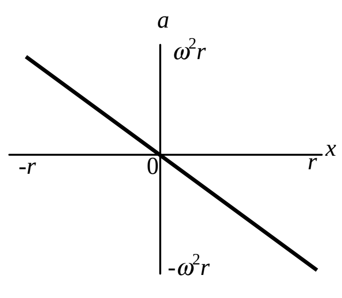
acceleration



Points to note

- The speed graph is the gradient of the displacement graph, and they have a phase difference of $\pi/2$ (= 90^o or a quarter of a period).
- The acceleration graph is the gradient of the speed graph, and is $\pi/2$ out of phase with the speed, and π out of phase with displacement.
- Each of these graphs has the same period.
- The acceleration is always opposite in sign to the displacement

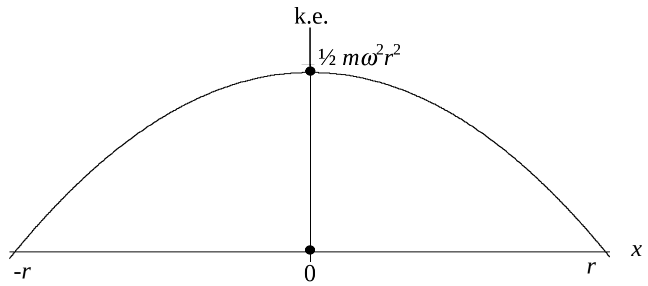
Acceleration against Displacement



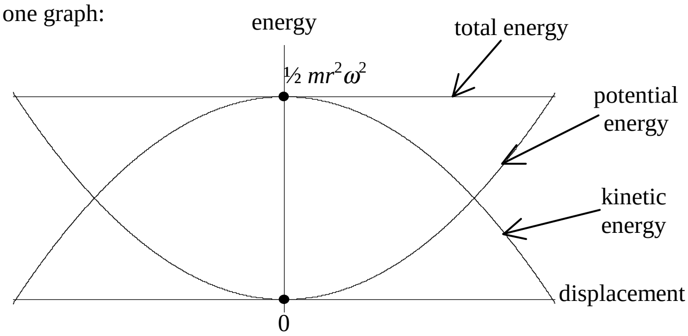
Energy Graphs

Energy against displacement
We will consider total energy, kinetic energy and potential energy.

Since total energy is constant, this graph is simply a horizontal line.
Kinetic energy, however, is more interesting:
We know that $v^2 = \omega^2(r^2 - x^2)$
So kinetic energy = $\frac{1}{2} mv^2 = \frac{1}{2} m\omega^2(r^2 - x^2)$
This leads to the graph below:

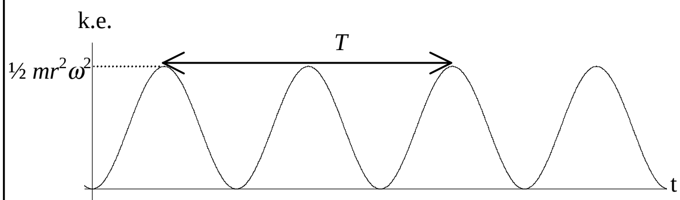


We can combine kinetic energy, potential energy and total energy on one graph:



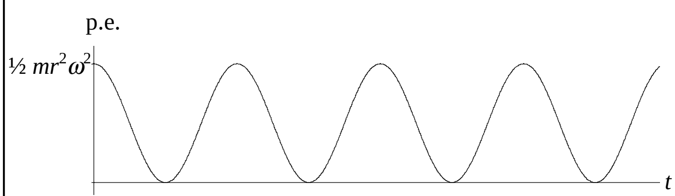
Note that the kinetic energy and potential energy at any time add up to the total energy, which is constant.

Energy against time
Since we know $v = -r\omega\sin\omega t$, we can deduce
$$\text{k.e.} = \tfrac{1}{2}mv^2 = \tfrac{1}{2} mr^2\omega^2\sin^2\omega t$$
This produces the following graph:



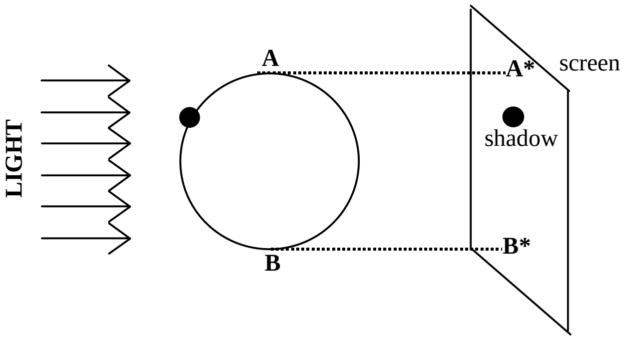
Note that the kinetic energy goes through **two** cycles during one period of the oscillation.

The total energy is, again, a constant, producing a horizontal straight line graph. The potential energy graph is an "upside down" version of the kinetic energy:

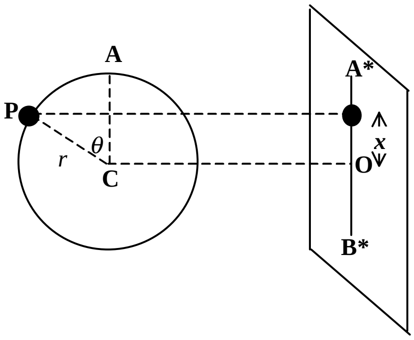


Circular Motion and SHM

To see the link between circular motion and SHM, consider an object performing horizontal circular motion at constant speed in front of a screen, with the plane of the motion perpendicular to the screen. Light is then directed onto the object, so that the object's shadow falls onto the screen. The diagram below shows a view looking down on the apparatus.



The shadow's motion will be in a straight line, between points A* and B*. In fact, this motion is simple harmonic; to show this we will need to introduce some angles and simple trigonometry.



- We will measure the displacement of the shadow from the point O, which is midway between points A and B. It is level with the centre of the circle, C. This corresponds to the equilibrium point in SHM.
- We will assume the object starts its circular motion at point A. This means its shadow will start at point A*. This corresponds to our assumption that SHM starts at a point of maximum displacement.
- The angle the line CP makes an angle θ with the line CA at time t.
- The particle is moving with a constant angular velocity ω.

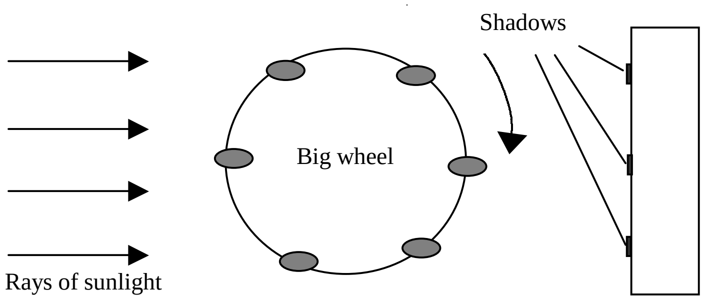
We now need to find an expression for the displacement, x, of the shadow at time t.

By trigonometry, $x = r\cos\theta$, where r = radius of the circle.
Since the particle has a constant angular velocity, $\theta = \omega t$
So at time t, displacement of shadow is given by
$x = r\cos\omega t$
This is therefore SHM.

Any SHM can have circular motion linked with it in this way – it is known as **associated circular motion.** The SHM is sometimes described as a **projection** of the circular motion. Note that ω in the SHM corresponds to the angular velocity in the associated circular motion

Exam Hint: - This only works if the circular motion is at <u>constant angular velocity</u> – so in many cases it will not apply to vertical circular motion.

Typical Exam Question
A large fairground wheel, which rotates at a constant rate, casts a shadow on to a nearby building. At a time when the sun's rays strike the building horizontally, a boy measures the speed of the shadow of one of the cars on the wheel as it passes different floors of the building.



At a floor which is level with the centre of the wheel, the speed of the shadow is 0.17 ms^{-1}. At a floor 10m higher, the speed is 0.16 ms^{-1}.
Calculate:
(a) the time it takes to complete one rotation; [6]
(b) the diameter of the wheel. [2]

(a) Shadow is projection of circular motion ⇒ SHM
Level with centre of wheel ⇒ speed is maximum.
So $0.17 = \omega r$ ✓

When $x = 10$ m, $v = 0.16$ms^{-1}. *So using* $v^2 = \omega^2(r^2 - x^2)$:
$0.16^2 = \omega^2(r^2 - 10^2)$ ✓
$0.16^2 = \omega^2 r^2 - \omega^2 10^2$ ✓

But $\omega^2 r^2 = (\omega r)^2 = 0.17^2$
So $0.16^2 = 0.17^2 - 100\omega^2$ ✓
$100\omega^2 = 0.17^2 - 0.16^2 = 0.0033$
$\omega^2 = 3.3 \times 10^{-5}$
$\omega = 5.7 \times 10^{-3}$ rad s^{-1} ✓

$T = 2\pi/\omega = 1.1 \times 10^2$ s ✓

(b) This is 2r.
$0.17 = \omega r \Rightarrow r = 0.17/\omega = 29.8$ m ✓
Diameter = 60 m (2SF) ✓

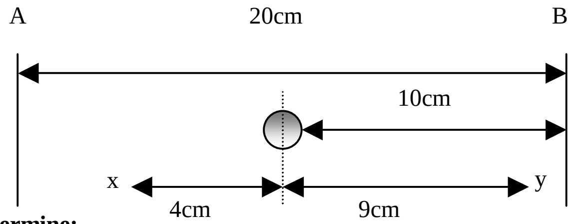
Questions

1. Explain what is meant by simple harmonic motion.
2. Write down expressions for the maximum speed and acceleration of a particle carrying out SHM.
3. Sketch graphs to illustrate displacement, speed and acceleration against time for a particle carrying out SHM. State the relationship between these graphs.
4. Sketch a graph to show how kinetic energy varies with displacement or a particle carrying out SHM. Include on your sketch the maximum value of the kinetic energy.
5. Explain why a projection of circular motion will not produce simple harmonic motion if the angular velocity is not constant.
6. A particle carrying out SHM has a period of 2.0 s and a maximum speed of 9.4 ms^{-1}. Given that timing starts when the particle's displacement is at a maximum, find the first 3 times when it is at a distance of 1.5 m from its equilibrium position.
7. A body carries out SHM. When its displacement from its equilibrium position is 0.10 m, its speed is 5.0 ms^{-1}. When its displacement from equilibrium is 0.30 m, its speed is 2.0 ms^{-1}.
   Calculate its maximum displacement.

Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

The spherical object shown in the diagram below is known to perform simple harmonic motion between points A and B. The object appears stationary when viewed with a strobe light at 21 Hz and 28 Hz but at no frequencies in between.



Determine:

(a) the frequency of the motion. **[1]**
28Hz ✗ *0/1*

> The student clearly did not understand what was happening with the strobe frequencies, but giving some answer, rather than no answer, was sensible, since it allows the student to continue with the question and hence gain some marks.

(b) the maximum speed of the object. **[2]**
$\omega = 2\pi f = 56\pi$ ✓
$v = \omega r = 56\pi \times 20 = 3520\ ms^{-1}$ *(3SF)* ✗ *1/2*

> The student gains the mark for using his/her value of frequency to find ω, but then uses both the wrong value for r (remember amplitude is half the "peak-to-peak" value – this is commonly examined!) and the wrong units (cm rather than m)

(c) the acceleration at position x. **[2]**
$\omega^2 x = 56\pi^2 \times 0.04$ ✓ $= 31000 ms^{-2}$ *(3SF)* *1/2*

> Again the student gains credit for using his/her own value of frequency – and here, s/he has remembered to work in SI units. But the student has forgotten that acceleration is a vector – it must have a direction! The words "towards the centre" or the use of a minus sign would have completed the answer.
> Also, although the student did not make this error in calculation, it was unwise to write $56\pi^2$ when $(56\pi)^2$ is meant, since it may lead to making the mistake of squaring just the π.

(d) the speed at point y. **[2]**
$v^2 = \omega^2(r^2 - x^2)$ ✓
$v^2 = 56\pi^2(20^2 - 9^2)$
$v = 9870000\ ms^{-1}$ ✗ *1/2*

> *Since the student has shown a suitable method, one mark can be awarded. However, s/he has forgotten to take the square root! The size of the answer should have alerted him/her to something wrong.*

Examiner's Answer

(a) The frequency must be a common factor of 21 and 28Hz i.e. 7Hz. ✓
(b) $\omega = 2\pi f = 14\pi$. ✓ $v_{max} = r\omega = 0.1 \times 14\pi = 4.4 ms^{-1}$ ✓
(c) $a = \omega^2 x = 14^2\pi^2 \times 0.04 = 77 ms^{-2}$ ✓ *towards the centre* ✓
(d) $v^2 = \omega^2(r^2 - x^2)$ ✓
$v^2 = 14^2\pi^2((0.1)^2 - (0.09)^2)$
v = 1.9ms^{-1} ✓

Answers

1 – 4 can be found in the text

5. If the angular velocity of the circular motion is not constant, then the angle through which the body turns will not be directly proportional to time. Accordingly, the displacement of the projection will not be proportional to the $\cos\omega t$, when ω is constant.

6. $T = 2\text{ s} \Rightarrow \omega = 2\pi/T = \pi \text{ rad s}^{-1}$
$v_{max} = \omega r = 3\pi \Rightarrow r = 3.0\text{ m}$
$x = 3.0\cos\pi t$
Distance of 1.5m from equilibrium $\Rightarrow x = \pm 1.5$ m
x = 1.5 m ⇒ $1.5 = 3.0\cos\pi t$
$0.50 = \cos\pi t$
$1.047 = \pi t$
$t = 0.33$ s

x = -1.5 m ⇒ $-1.5 = 3\cos\pi t$
$-0.50 = \cos\pi t$
$2.094 = \pi t$
$t = 0.67$ s

We know the body takes half a period (= 1s) to travel from one extreme displacement to the other.

So to travel from $x = -1.5$ m to the negative extreme takes (1.0 – 0.67) = 0.33 s
It then takes a further 0.33s to return to $x = -1.5$ m

So the first 3 times are: 0.33s; 0.67s; 1.33s

7. $v^2 = \omega^2(r^2 - x^2)$
$5^2 = \omega^2(r^2 - 0.1^2)$ ①
$2^2 = \omega^2(r^2 - 0.3^2)$ ②

① ÷②: $\frac{5^2}{2^2} = \frac{\omega^2(r^2 - 0.1^2)}{\omega^2(r^2 - 0.3^2)} = \frac{(r^2 - 0.1^2)}{(r^2 - 0.3^2)}$
$\frac{25}{4} = \frac{(r^2 - 0.01)}{(r^2 - 0.09)}$
$25(r^2 - 0.09) = 4(r^2 - 0.01)$
$25r^2 - 2.25 = 4r^2 - 0.04$
$21r^2 = 2.21$
$r^2 = 0.105$
$r = 0.32$ m (2SF)

> When you write down your final answer you are expected to use the same number of significant figures as the data that you were given. In mid calculation it doesn't really matter if you use one more significant figure because it is your method rather than your mid-way result that is being marked.
>
> The same thing applies to units. You may choose to leave out units in the middle of calculations but you **must** include them with your final answer.

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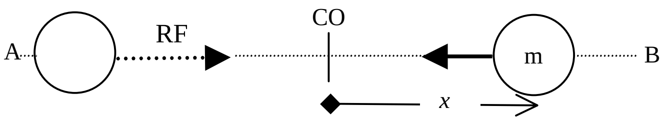
Physics Factsheet

April 2003 Number 53

Mechanical Oscillations & the Mass-spring System

This Factsheet covers the applications of simple harmonic motion (SHM) to mechanical oscillations. Factsheet 20 covers the basics of SHM. Factsheet 54 will cover the simple pendulum.

- You need to be able to recognise SHM from the forces acting. The diagram below shows a mass *m* oscillating along the line AB about the centre of oscillation (CO)
  No matter which way the mass is moving, the force acting is **always directed to CO.** Because it always tries to return the mass to the centre, it is called a *restoring force* (RF).



- For SHM, the restoring force is **directly proportional to the displacement** (x) from the **centre of oscillation**. RF $\propto x$
  So the further away the mass is from CO, the larger the size of the force on it.

So to test for SHM, we must analyse the forces acting and check that these two conditions apply:

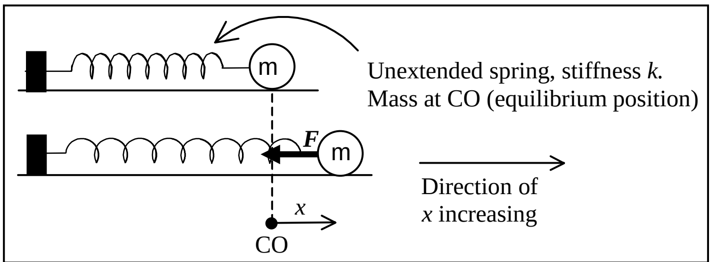
(i) there must be a restoring force acting. (ie a force acting towards a fixed point so as to reduce x)
(ii) the restoring force must be directly proportional to the displacement.

When x is positive the force F is in the negative direction. The opposite is true when x is negative. This means that the two conditions can be stated mathematically as

$$F \propto -x \quad or \quad F = -(constant) \times x \quad ①$$

Example : A simple mass-spring system
This is one of the key examples of SHM that you need to know.

A mass m rests on a horizontal frictionless surface and is connected to a spring with stiffness constant k.
The tension in the spring and the extension are linked by Hooke's law:
Tension = k × extension



When the mass is displaced as shown, there is a force F acting on the mass where F = tension in the spring. Hooke's law tells us the size of the force: $F = k \times x$.

But since F and x are vectors, we must measure them in the **same** direction.

The rule is that always work in the direction of x increasing

F is in the opposite direction to x increasing, so we have $F = -kx$.
So the restoring force is directly proportional to x - so, when the mass is released, it will move with simple harmonic motion.

Exam Hint: *You must be careful with x. We often use x for the extension of a spring and also for the displacement of a body. In this example, they are the same but this is often not the case. As a rule, use X for the displacement and then figure out the extension.*

The next stage in looking at SHM is to use Newton's 2nd law of motion in the form $F = m \times a$. Again this is a vector equation with both F and a being in the same direction. So, looking in the direction of x increasing:

$$F = m \times a \qquad -kx = m \times a \qquad a = -\left(\frac{k}{m}\right)x$$

This last equation is now compared with a standard equation for SHM:
$a = -\omega^2 x$

So $\omega^2 = \frac{k}{m} \qquad \omega = \sqrt{\frac{k}{m}}$

We use ω to find the period, frequency, velocity & displacement.
For example:

Period $T = \frac{2\pi}{\omega}$, so here, $T = \frac{2\pi}{\omega} \Rightarrow T = \frac{2\pi}{\sqrt{\frac{k}{m}}} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$

The formula for the periodic time is usually given on a 'formula sheet' so you need not commit it to memory. It is worth noting that the period increases with the mass. Mass is a measure of inertia, it tells you how sluggish the body is to movement. A really massive body would take longer to oscillate ie, its period would increase. Because k is in the bottom line, the opposite effect takes place. If a spring is 'lazy', it is easy to extend and k is small. Think of a bungee rope. k is small but '1 over a small number is large' and so the period is 'large'.

Summary. In dealing with SHM questions first draw a sketch showing the forces acting together with a clear indication of x (the displacement) increasing. Then, (in this case) from a knowledge of springs deduce that, $F = -k \times x$. This demonstrates SHM. The next stage is to find the corresponding equation for acceleration and compare it with $a = -\omega^2 x$ to deduce ω. Almost all SHM quantities are determined by ω. One important exception is the amplitude. This is determined by something specific to the question such as initial displacement or velocity. It is not determined by the mass of the body nor the stiffness of the spring.

Example: Vertical mass-spring system

In this example, we also have to consider the effect of the weight of the mass, since the spring would not be unextended at equilibrium. The key ideas are:

- First consider the mass at equilibrium – this allows you to find any unknown quantities, such as the spring constant or the extension at equilibrium
- When showing the motion is SHM, take x to be the **downwards** displacement from the **equilibrium position** (CO). So in this case, the extension of the spring will be x + *extension at equilibrium*
- As with the horizontal mass-spring system, we have $\omega = \sqrt{\frac{k}{m}}$;

  unless you are asked to show the motion is simple harmonic, you can quote this relationship

An elastic cord has an unstretched length of 30cm. A mass of 60 grams is attached to one end and hangs freely from the cord whose other end is attached to a fixed support. The mass then extends the length of the cord to 40cm.. Assume that the cord obeys Hooke's law.

(a) Calculate the 'spring constant' for the cord.

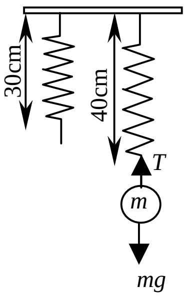
The mass is then displaced 5.0cm vertically downwards and released from rest

(b) Show that its motion is simple harmonic

(c) Find:
- **(i) the periodic time for one oscillation**
- **(ii) the amplitude of the motion A**
- **(iii) the greatest speed of the mass and state where it occurs**
- **(iv) the greatest acceleration of the mass and state where it occurs**

(take the acceleration of free fall, g to be = 10 ms^{-2})

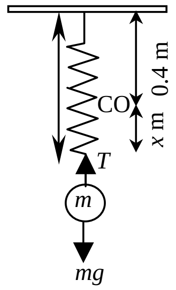
(a)



Condition for equilibrium is $T = mg$.
Converting mass into kg and extension into m:

$$k = \frac{mg}{\text{extension}} = \frac{60\times10^{-3}\times10}{10\times10^{-2}} = 6\text{Nm}^{-1}$$

(b)



We measure everything **downwards** (as x is measured downwards)
Resultant force on m is $mg - T$

But $T = k \times$ *extension*
Extension = $0.1 + x$ (since it is x m below CO)
So $T = k(0.1 + x) = 6(0.1 + x)$

So resultant force $F = (0.06)(10) - 6(0.1 + x)$
$= 0.6 - 0.6 - 6x = -6x$
So motion is simple harmonic

(c) (i) From (b) we have $-6x = 0.06a$, so $a = -100x$
Hence $\omega^2 = 100$ and $\omega = 10$

So $T = \frac{2\pi}{\omega} = \frac{2\pi}{10} = 0.63$s (2SF)

(Or you could have done this using the standard formula for ω^2)

(ii) We are told 'the mass is displaced 5cm vertically downwards'. This then is the amplitude in this case.
When released, the mass will rise 5cm to the equilibrium position and then another 5cm above the equilibrium position.
So A = 0.05m.

(iii) The maximum speed occurs at the centre of oscillation.
$V = A\omega$
$= 5 \times 10^{-2} \times 10$
$= 0.5 \text{ ms}^{-1}$

(iv) Remember that the defining equation for SHM is:- $a = -\omega^2 x$
It follows that the greatest acceleration occurs when x has its greatest value.
This is clearly when $x = \pm A$ ie at the points when the mass is at its highest and lowest points.
In this case, $\omega = 10 \text{ s}^{-1}$ and $A = 0.05$ m.
The numerical value of the greatest acceleration is
$(10)^2 \times (0.05) = 5\text{ms}^{-2}$

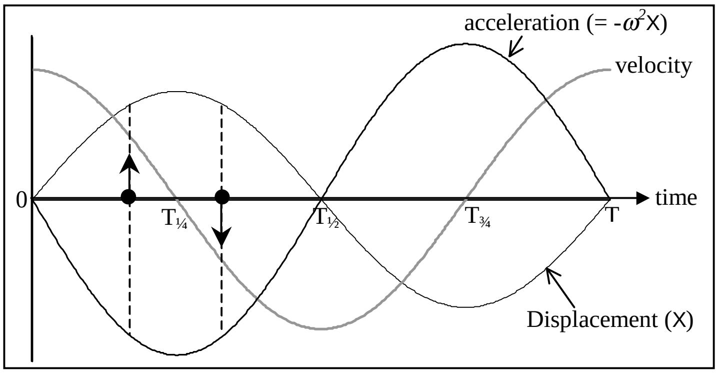
Displacement, Velocity, Acceleration and Phase

The graphs (right) show the relationship between the displacement (x), the velocity (v) and the acceleration (a) for a mass-spring system. If you start with the displacement, the gradient of its graph against time (at any position) will give you the value of the velocity at that point. The gradient of the velocity against time graph will give you the value of the acceleration.

The three graphs are all sinusoidal (take sine or cosine shapes) and as such are often drawn with angles on the x-axis.

The diagrams provide a good example to explain the term **'phase difference.'**

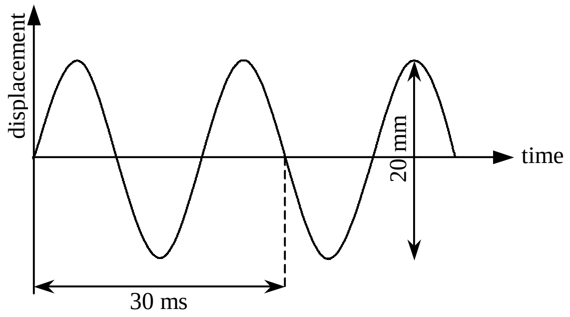
Look at the displacement and velocity in fig 1. They are 'out of step' with each other or, out of phase. The phase difference here is 90° (or $\pi/2$) .The phase difference between displacement and acceleration is 180° (or π).



Exam Workshop

This is a typical weak student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

A small machine part is vibrating with simple harmonic motion; its displacement / time graph is shown below.



For the machine part,

(i) state the amplitude of the motion **[1]**

The amplitude a = 10mm ✓ *1/1*

> This is correct. The student has halved the peak to peak value and got the units correct. Beware that at a later stage, this may need converting to metres.

(ii) calculate the periodic time **[1]**

The periodic time $T = 30\ ms = 30 \times 10^{-3}\ s$ ✗ *0/1*

> This is incorrect; the student may have rushed the question. 30 ms is for **three** half cycles, not two An error like this will make subsequent calculations wrong but the student is only penalised once. The examiner will 're-work' the rest of the question as if T were 30ms. For this reason you should explain your working as fully as possible and write equations before substituting values.

(iii) the frequency (*f*) **[1]**

$$frequency = \frac{1}{time\ period} = \frac{1}{30\times10^{-3}} = \frac{10^3}{30} = 33.3Hz$$ ✓ecf *1/1*

> The student has shown the work required. The **e**rror **c**arried **f**orward has been allowed for.

(iv) the angular speed (ω) **[1]**

$\omega = 2\pi f$

$\omega = 2\pi \times 33.3 = 209\ s^{-1}$ ✓ecf *1/1*

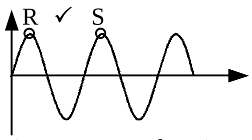
> Again the student has shown working (incorrect *f* carried forward). Correct answer. Note, ω measured in s^{-1}

(v) the maximum speed, and label any two points (P &Q) on the graph where this occurs. **[3]**

$v = \omega\sqrt{A^2 - x^2}$ ✗✗ *0/3*

> The student has reproduced a formula but cannot apply it. No points shown on graph.

(vi) the maximum acceleration, and label any two points (R & S) where this occurs. **[3]**



Maximum acceleration is at extreme of motion when x = amplitude.

$a = -\omega^2 x = -209^2 \times 10 = 436810\ ms^{-1}$ ✓ ✗ *2/3*

> Points R & S correct The student has used the correct formula, (allowance for incorrect ω =209) but has **incorrect units.**

If the machine part has a mass of 20 grams, calculate

(vii) the greatest restoring force acting on it. **[1]**

F = ma

$F = 20\times10^{-3} \times 436810$ *Newtons.* ✓*ecf* *1/1*

> The student has quoted the correct formula and substituted what s/he thinks the correct values.
> This example illustrates the comparatively large forces acting on the body (about 100 × weight) when vibrating.

(viii) Without further calculation, comment on the resultant force acting on the mass at the points you have labelled P & Q. **[1]**

0/1

> The student did not answer section v) but, the speed is a *maximum*, ie not changing which means the acceleration at points P & Q is zero. Hence the resultant force at these points is zero

Examiner's Answers

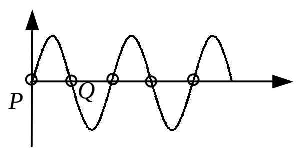
(i) The amplitude a = 10mm (= 0.01 m) ✓

(ii) 3 half cycles in 30 ms
2 half cycles in 20 ms
period = 20 ms ✓

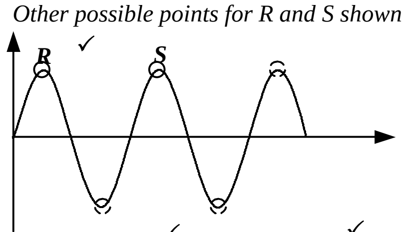
(iii) $f = \frac{1}{20\times10^{-3}} = 50Hz$ ✓

(iv) $\omega = 2\pi f$
$\omega = 2\pi \times 50 = 100\ \pi = 314\ s^{-1}$ ✓

(v) The maximum speed is when:
$x = 0 \Rightarrow V_{max} = \omega A = 314 \times 10 \times 10^{-3} = 3.14\ ms^{-1}$ ✓
One mark for correct units.
Points for P and Q shown occurs when gradient of displacement/time graph greatest(+ or -). Any two✓



(vi) 1 mark for correct position of R and S
Other possible points for R and S shown dotted



$a_{max} = \omega^2 \times A = 314^2 \times 10 \times 10^{-3} = 986\ ms^{-2}$ ✓ ✓

(vii) $F = ma = 20\times10^{-3}\times 986 = 19.7\ N$

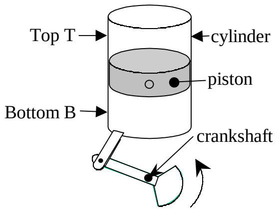
(viii) the speed is a maximum , ie, not changing which means the acceleration at points P & Q is zero. Hence the resultant force at these points is zero✓

Exam Hint: *In answering SHM questions, focus on ω. Without this, not much else can be found. It is often supplied via periodic time $T = \frac{2\pi}{\omega}$ or the frequency $f = 2\pi\omega$. Remember that ω is constant for a particular mass and spring $\left(\omega = \sqrt{\frac{k}{m}}\right)$ but the amplitude can be set at any sensible value. For example in the question above, the amplitude could have been set at 2cm. (so changing greatest speed and acceleration) but* ***ω would not change.***

- *Do not confuse a for acceleration and A for amplitude.*
- *A lot of numerical questions involve grams and centimetres, kilograms and metres. Be ready to convert!*

Typical Exam Question

The diagram represents the cylinder and piston of a car engine. As the piston moves from position T down to B and back to T again, the crankshaft rotates through 360°. The distance BT is 10 cm. and the crank speed is 3000 revolutions per minute. For the piston, find:



(i) its frequency;

(ii) its amplitude.

You may assume that the motion of the piston is simple harmonic. Find

(iii) the maximum speed of the piston, stating its position where this occurs,

(iv) the maximum acceleration of the piston, stating the position where this occurs.

Answers:

(i) $f = \frac{3000}{60} = 50$ Hz

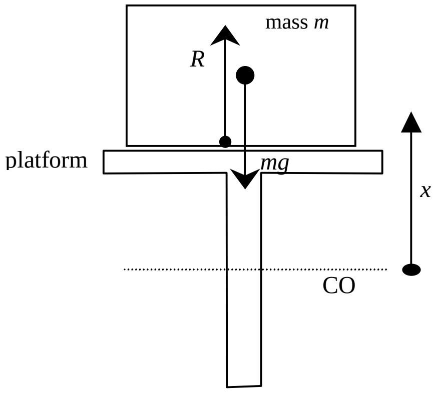
(ii) amplitude $= \frac{BT}{2} = \frac{10\,cm}{2} = 5cm = 5 \times 10^{-2}m$

(iii) $V_{max} = A \times \omega = A \times 2\pi f = 15.7$ m s^{-1} . Midpoint of BT.

(iv) $a = -\omega^2 x$
$a_{max} = \omega^2 \times A = 4930ms^{-2}$. At both points B and T

Questions

1. The diagram shows a mass *m* resting on a platform that can oscillate with SHM in a vertical direction. The mass is shown above the centre of oscillation with the forces acting on it. *R* is the reaction between the platform and the mass.



(i) Write down the equation of motion for the mass whilst it remains on the platform.

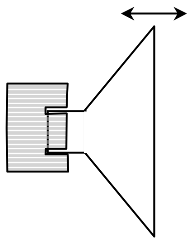
(ii) What is the acceleration of mass/platform when the mass loses contact with the platform?

(iii) Hence deduce the relationship between frequency and displacement when contact is lost.

(iv) If the platform has a fixed frequency of 10 Hz, find the amplitude when contact is just lost.

(v) If the platform has a fixed amplitude of 1.0 mm, find the frequency when contact is just lost.

2. A loudspeaker produces a musical note by the oscillation of a diaphragm. If the amplitude is 1.0×10^{-3} mm, what frequencies are available if the diaphragm's acceleration must not exceed 10 ms^{-2} ?



3. A block of mass of 600 grams is hung from a spring. It performs vertical oscillations with a periodic time of 0.5s.
(i) Calculate the spring constant.

An additional mass of 200 grams is now added to the block.
(ii) Calculate the new periodic time.

Answers

1. (i) $(R - mg) = m \times a. \Rightarrow R = m(g+a)$
(ii) contact lost when R=0, ie a= -g
Or the acceleration is g toward the CO
(iii) $a = -\omega^2 x \Rightarrow -g = -\omega^2 x.$ $(g = \omega^2 x \Rightarrow g = 4\pi^2 f^2 x)$
(iv) $9.8 = 4\pi^2\, 10^2\, x \Rightarrow x = 2.5mm$
(v) $9.8 = 4\pi^2 f^2\, (1.0 \times 10^{-3}) \Rightarrow f = 16$ Hz (2SF)

2. For SHM $a = -\omega^2 x$
Numerical values only:
$\omega^2 x \le 10$ or $4\pi^2 f^2 x \le 10 \Rightarrow f \le 500Hz$ for $x = 10^{-6}m$

3. (i) $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow 0.5 = 2\pi\sqrt{\frac{0.6}{k}} \Rightarrow k = 95\ Nm^{-1}$
(ii) $T = 2\pi\sqrt{\frac{0.6 + 0.2}{94.7}} \Rightarrow T = 0.58$ s (2s.f.)

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Physics Factsheet

April 2003 Number 54

The Simple Pendulum

This Factsheet follows on from Factsheet 20 (Simple Harmonic Motion) and Factsheet 53 (Mechanical Oscillations).

The simple pendulum is an example of a system that oscillates with simple harmonic motion(SHM). It consists of a small mass (a bob) attached to a thin thread and allowed to oscillate. The centre of oscillation is the position of the bob when the thread is vertical.

There are two possible ways of measuring the displacement of the mass m from the centre of oscillation: x (distance) or θ (angle) as shown below. θ must be measured in **radians** (see box right). θ and x are related by: $\theta = \frac{x}{\ell}$

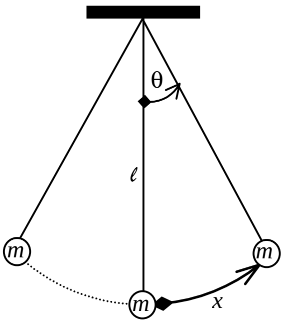


Fig 1a

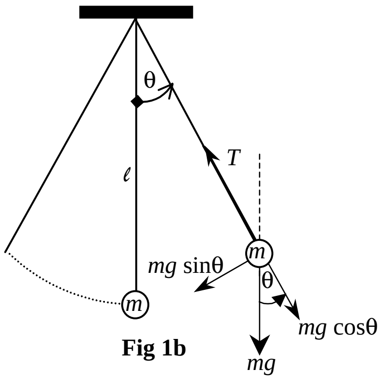


Fig 1b

Fig 1b shows how to analyse the forces acting on the bob.
The two forces involved are the weight mg (acting vertically) and the tension T acting along the thread.
We resolve the forces along and perpendicular to the thread.

Along the thread, no motion takes place, so we have

$T = mg\cos\theta$

At right-angles to the thread, in the direction of increasing θ, we have:

Resultant force $= -mg\sin\theta$

The angle θ is *small* and measured in *radians,* so $\sin\theta \approx \theta$.

So resultant force $= -mg\sin\theta$
$= -mg\theta$ (using the above approximation)
$= -mg\frac{x}{\ell}$

We now have the requirement for SHM i.e. the restoring force is proportional to the displacement x

To complete the proof, we use Newton's second law ($F = ma$) in the direction of x increasing:

$F = ma$

$-mg\frac{x}{\ell} = ma$

$a = -\frac{g}{\ell}x$ This is of the form $a = -\omega^2 x$, - the standard equation for SHM.

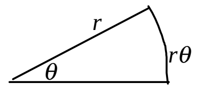
So we have $\omega^2 = \frac{g}{\ell}$, and so $\omega = \sqrt{\frac{g}{\ell}}$

Radians
Radians are an another way of measuring angles.

radians = degrees $\times\frac{\pi}{180}$ degrees = radians $\times\frac{180}{\pi}$

Approximations: If θ is a small angle, measured in radians, then: $\sin\theta \approx \theta$ $\cos\theta \approx 1$ $\tan\theta \approx \theta$

Arcs: The length of an arc of a circle is given by $r\theta$, where r is the radius of the circle and θ the angle subtended by the arc.



Period of a pendulum

Since $T = \frac{2\pi}{\omega}$ and $\omega = \sqrt{\frac{g}{\ell}}$, $T = \frac{2\pi}{\sqrt{\frac{g}{\ell}}} = 2\pi\sqrt{\frac{\ell}{g}}$

The most striking aspect of this equation is that the time period is independent of the bob's mass m. Changing the mass **will not** change the period. The only way to alter the period of a simple pendulum is to alter its length ℓ.

Another point worth noting is the units in the equation $T = 2\pi\sqrt{\frac{\ell}{g}}$

The units of the left hand side must be equivalent to those on the right. Writing m for metres and s for seconds, the units for acceleration are ms^{-2}.

So the units of $2\pi\sqrt{\frac{\ell}{g}}$ are $\sqrt{\frac{m}{\left(\frac{m}{s^2}\right)}} = \sqrt{s^2} = s$

The unit for T is s, so the equation is consistent in terms of units.

Worked Example: ***A long case (grandfather) clock has a pendulum. To a good approximation, we may consider it to be a 'simple pendulum'. The bob may be moved up or down by means of an adjusting nut.***

(a) If the clock runs too fast, which way should the bob be moved, up or down? Explain your answer.

(b) The clock 'beats seconds.' That is, each half swing, a tick or a tock, is one second. Find the length of the simple pendulum required if the acceleration due to gravity is 9.80 ms⁻².

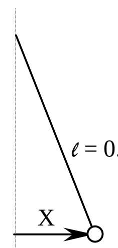
(a) If the clock is running too fast (gaining), then the pendulum is oscillating too quickly - its period (T) is too small. So T must be increased and since $T = 2\pi\sqrt{\frac{\ell}{g}}$ *we must increase the length* ℓ *- the bob should be moved down*

(b)Each half swing is one second ⇒ *full oscillation takes 2 seconds*

$2 = 2\pi\sqrt{\frac{\ell}{9.80}}$ so $\frac{1}{\pi} = \sqrt{\frac{\ell}{9.80}}$

$\ell = \frac{9.80}{\pi^2} = 0.993m$

Typical exam question
The diagram shows a simple pendulum with a length of 0.90m.
The pendulum bob is drawn aside as shown through a distance X.



(a) If the distance X is 10 cm, calculate, for the bob,
- **(i) the periodic time**
- **(ii) the initial acceleration of the bob**
- **(iii) the subsequent maximum speed of the bob.**

(b) A second mass is now held on the vertical passing through the pendulum's point of suspension. Where must this second mass be placed so that when both masses are *released simultaneously*, they will collide with each other?

(c) If the distance X is increased to 15cm, will the two masses again collide? Explain your answer. You may take the acceleration of gravity to be 9.8 ms^{-2}.

Answer

(a) (i) $T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{0.9}{9.8}} = 1.9s$

Now use this to find ω. $\omega = \frac{2\pi}{T} = \frac{2\pi}{1.9}$ $\omega = 3.3s^{-1}$.

(ii) for SHM $a = -\omega^2 x$. *Released when x = 0.1m.*
initial acceleration $= (3.3)^2 \times 0.1 = 1.1\ ms^{-2}$ *(2s.f.)*

(iii) $V_{max} = A\omega = 0.1 \times 3.3 = 0.33ms^{-1}$

(b) Time for bob to reach vertical is $\frac{T}{4}$ *seconds* = $\frac{1.9}{4}$ = 0.475 s.

During this time, second mass falls vertically under gravity.
Use $s = ut + ½ g t^2$
$s = 0 + ½ \times 9.8 \times (0.475)^2 = 1.1\ m$

(c) The second mass must be placed 1.1 m above lowest position of bob ie 0.2 m above point of suspension. If the distance X is increased to 15 cm, the period of the pendulum will ***not change.*** *It will still take 0.475 s to cover the increased distance. Hence the masses will still collide.*

Exam Hint:
1 *Most SHM questions require a value for ω. A common way of supplying ω is through the periodic time T. Remember from your SHM work that ω is radians per second, 2π is radians and T is seconds so it is easy to quote:*

$$\omega = \frac{2\pi}{T}$$

2 *This question supplies numerical values to two significant figures. This means that final answers should be to two significant figures; however, it always a good idea to work to more than this until stating the final answer. In this case, as an intermediate step, we find a quarter period to 3 sig fig (0.475 s). Writing 0.48 may give an inaccurate answer in later parts of the question.*

Using a pendulum to find the acceleration of gravity

The acceleration of gravity can be found with a simple pendulum. We know that the period is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$.

So, by measuring T and ℓ we should be able to figure out a value for g.

Squaring the above equation gives $T^2 = 4\pi^2\frac{\ell}{g}$.

In this equation, T and ℓ are the variables whilst π and g are constants.

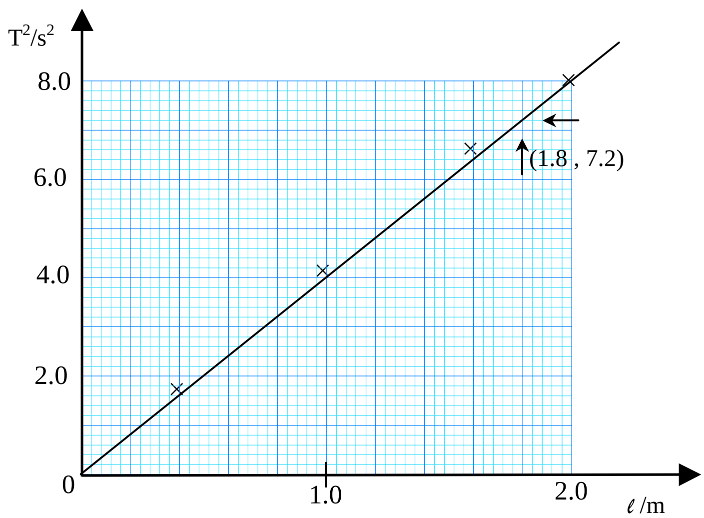
Re-writing the equation with the constants in a bracket: $T^2 = \left(\frac{4\pi^2}{g}\right) \times \ell$.

This is the same form as the equation of a straight line. So, a graph of T^2 against ℓ will be a **straight line** with **gradient** $\frac{4\pi^2}{g}$

Table 1 shows in the first two columns the length and period of a simple pendulum. You would be expected to figure out the third column and complete it.

ℓ/m	T/s	T^2/s^2
0.4	1.27	1.61
1.0	2.00	4.00
1.6	2.55	6.50
2.0	2.82	7.95

The graph has been drawn for you and a suitable point (1.8, 7.2) identified for gradient calculation.



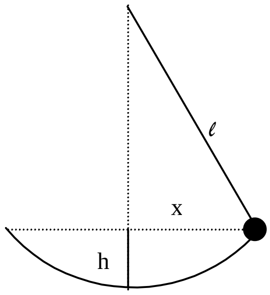
From the graph, the units for the gradient are s^2/m. This is 'the upside down' of acceleration units.

This is correct because the gradient is $\frac{4\pi^2}{g}$.

To find g we solve the equation: $\frac{7.2}{1.8} = \frac{4\pi^2}{g} \Rightarrow g = 4\pi^2 \times \frac{1.8}{7.2} = 9.87ms^{-2}$

Exam Workshop
This is a typical weak student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

The diagram shows a simple pendulum. The length of the pendulum, ℓ is 80cm and the mass of the bob is 100 grams.



In this question, take the acceleration of free fall to *be g = 9.8 ms^{-2}*
For the pendulum, calculate:
(i) the periodic time T **[2]**

$T = 2\pi\sqrt{\frac{\ell}{g}}$ ✓

$T = 2\pi\sqrt{\frac{80}{9.8}}$ ✗

$T = 2\pi \times 2.86$
$T = 18\ s$ 1/2

> The candidate has the correct formula but not changed centimetres into metres

(ii) the angular velocity ω **[2]**

$\omega = \frac{angle\ turned}{time\ taken} = \frac{2\pi}{T}$ ✓

$\omega = \frac{2\pi}{18} = 0.35\ rad\ s^{-1}$ **ecf** 2/2

> The candidate has used the correct formula and brought down an error; no penalty here. ecf=error carried forward.

The bob is now drawn aside horizontally through a distance X = 10cm and released from rest. For the subsequent motion of the bob, find
(iii) the amplitude **[1]**
The amplitude = 10 cm = 0.1 m ✓ 1/1

> Correct, the candidate has changed units,

(iv) the maximum velocity of the bob **[3]**
max velocity $= A \times \omega$ ✓
$= 0.1 \times 0.35$ **ecf**
$= 0.035\ ms^{-1}$ ✓ 3/3

> Full marks here, again an allowance for carrying an error forward.

(v) As the bob continues to oscillate, find the maximum kinetic energy of the bob and the position where this occurs. **[3]**
The maximum kinetic energy $= ½ \times mu^2$ ✓
$= ½ \times 0.1 \times 0.035^2$ **ecf** ✓
$= 0.000061\ J$ ∧ 2/3

> Again one mark for carrying an error forward, one for k.e. but omitted <u>position</u>

(vi) Hence find the maximum change in potential energy and deduce the vertical displacement h through which the bob moves. **[3]**
Potential energy mgh ∧ ∧ ∧ 0/3

> The candidate cannot get started. Not even one mark for p.e.; the change has not been equated to change of k.e

(vii) The bob is now drawn aside horizontally through a distance X = 15 cm and released from rest. Without further calculation, state and explain what will happen to the periodic time T and the maximum amounts of kinetic and potential energies. **[4]**
The period increases ✗ ∧ 1/4
The maximum energies increase ✓∧

> First part wrong, & no explanation. Second correct but lacks explanation.

Examiner's Answers

(i) $T = 2\pi\sqrt{\frac{\ell}{g}}$ ✓ $= T = 2\pi\sqrt{\frac{80}{9.8}}$ ✓ $= 1.8\ s$ *(s.f.)*

(ii) $\omega = \sqrt{\frac{g}{l}}$ ✓ $= \sqrt{\frac{9.8}{0.8}}$ ✓ $= 3.5\ s^{-1}$

(iii) *Amplitude = 10 cm = 0.10 m* ✓

(iv) $V_{max} = A\ x\ \omega = 0.1 \times 3.5 = 0.35\ ms^{-1}$ ✓ ✓ ✓

(v) $ke_{max} = \left(\frac{m}{2}\right)u^2 = \left(\frac{0.1}{2}\right) \times (0.35)^2 = 6.1 \times 10^{-3} J$ *(at lowest point)* ✓ ✓ ✓

(vi) $\Delta(k.e)^2 = \Delta(p.e) \Rightarrow 6.1 \times 10^{-3} J = mgh \Rightarrow h = 6.3\ mm\ (2s.f)$ ✓ ✓ ✓

The period is independent of amplitude (determined only by l and g) and remains constant. The maximum k.e. and p.e. both increase. ✓ ✓

Questions

1. Write down the equation for the period of a simple pendulum.

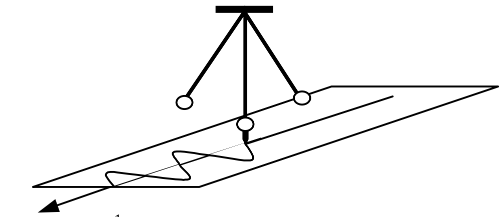
2. Write down the equation for (a) the frequency and (b) the angular frequency (ω) in terms of the acceleration of gravity and the length of the pendulum.

3. Explain how you would accurately measure the period of a simple pendulum.

4. In an experiment to measure the acceleration of free fall using a simple pendulum
   (a) explain what measurements you would take;
   (b) state what quantities you would plot along the y and x axes of the graph that you would plot;
   (c) explain how you would use your graph to calculate the acceleration of free fall.

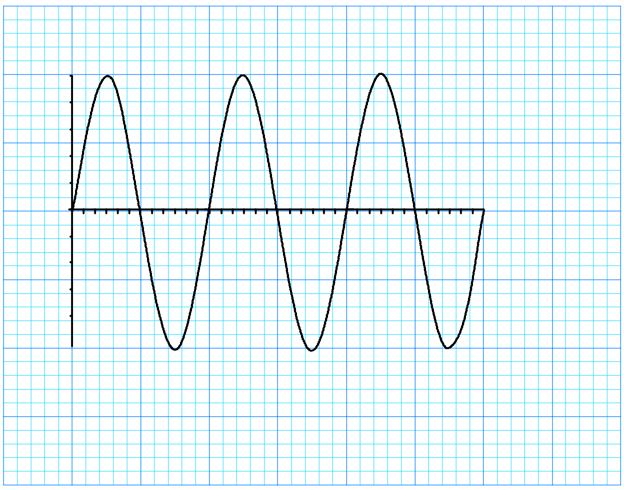
5. (a) A simple pendulum has a frequency of 1.5 Hz. Calculate
   (i) its periodic time
   (ii) the length of the pendulum (take g = 9.8 ms^{-2}).

   (b) The motion of the pendulum is recorded by a moving strip as shown.



   The strip moves uniformly with a speed u of 150 mm s^{-1}.By considering the number of oscillations in 1 second,
   (i) Find the "wavelength" of the trace obtained.
   (ii) The pendulum is now replaced by one with a frequency of 3.0Hz. Find the new "wavelength" obtained.

   (c) On the graph label and number the axes so as to give an accurate description of the original pendulum, given that it has an amplitude of 10mm.



   (d) Calculate:
   (i) the maximum speed of the pendulum bob and mark two points where these speeds are in opposite directions.
   (ii) the maximum acceleration of the bob and mark two points where these accelerations are in opposite directions.

Answers

1. $T = 2\pi\sqrt{\frac{\ell}{g}}$

2. (a) $f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$

(b) $\omega(= 2\pi f) = \sqrt{\frac{g}{\ell}}$

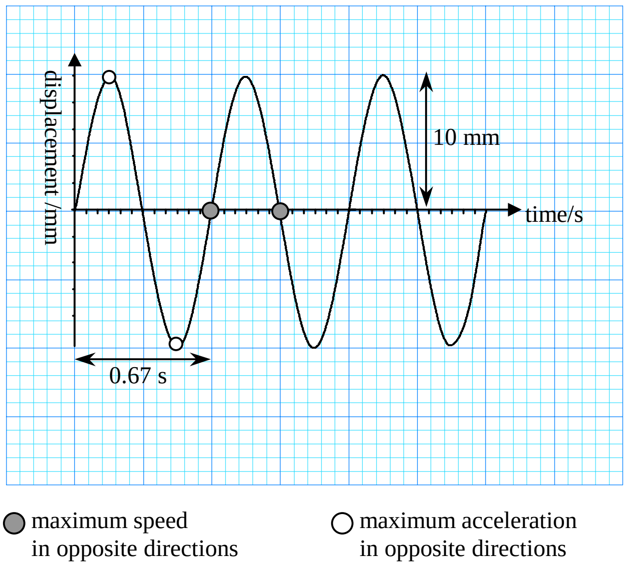
3. Choose the centre of the oscillation and place a fiducial mark, such as a pin held in a cork, at that place. Pull the pendulum aside and count down from 3 to zero each time the bob passes the fiducial mark, moving in the same direction. As the bob passes this mark on "zero" start the timer and time 20 oscillations. Repeat this procedure and average your value. Divide the mean time for 20 oscillations by 20 to get a value for the period.

4. See text.

5) (a) (i) 0.67 s
(ii) 11.0 cm

(b) (i) 100 mm
(ii) 50 mm

(c)



(d) (i) 9.4×10^{-2} ms^{-1}
(ii) 0.89 ms^{-2}

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Answering Exam Questions: Simple Harmonic Motion

This Factsheet looks at how to approach A2 simple harmonic motion (SHM) questions. A later Factsheet will look at questions on damped and forced oscillations, resonance and their applications. By the end of this Factsheet, you should be more confident about:
- What the examiners want
- The kinds of things you are likely to be asked
- Common mistakes and misunderstandings

Before starting on this Factsheet, you should be familiar with the basic concepts of SHM (see Factsheet 20)

What do you have to be able to do?
Questions will be assessing one or more of the following:
- Understanding of the conditions for SHM
- Knowing the terminology associated with SHM
- Using the equations associated with SHM
- Understanding energy exchange in SHM
- Drawing and/or interpreting graphs to represent the change with time of displacement, velocity, acceleration and energy
- Applying your knowledge of SHM to specific systems, such as the simple pendulum or mass-spring system

How should you approach revising this topic?
- Like all topics, some basic factual **learning** is required - for example, the **conditions** and **terminology** have to be memorised. For many specifications, the **equations**, including those for the simple pendulum and mass-spring system, appear on the **formula sheet** - but if they don't, they have to be learned.
- Carrying out **calculations** - once you know the equations - is largely a matter of **practice** - so the best approach here is to tackle plenty of examples.
- With **energy changes** and **graphs**, probably the best approach is a mix of understanding and learning: trying to derive these from scratch in the exam is only for the very confident, but straight rote learning without understanding is liable to create problems if the question changes things a little - for example, by changing the starting position of the object.

Conditions for SHM/ Definition of SHM
This can be in words or an equation, but if you use the equation, you must define and explain the symbols.

Word definition:
- *The acceleration (or force) is directed towards a fixed centre point. (or is directed towards the equilibrium position)*
- *The size of the acceleration (or force) is proportional to the distance (or displacement) from the fixed centre point.*

Equation definition:
- $a = -kx$
  where a = acceleration, x is displacement from a fixed centre point and k is a positive constant
- *the negative sign means the acceleration is directed back towards the fixed centre point.*

> ***Exam Hint** - There are usually two marks for this question - allocated according to the bullet points above.*
> *It is probably a better strategy to give the definition in words - as provided you remember "two marks mean two points" there is less opportunity to omit something crucial!*

SHM equations
Below is a list of equations you may encounter in SHM. Please note:
- There is a variation in symbols used between different boards - stick to the symbols you are used to!
- For some equations, there is more than one way to express them - again, this depends on the exam board.
- Some equations are "core" - everyone has to know them. Other "additional" equations may be useful, but they are not essential unless mentioned in your specification

Remember to check what is on your formula sheet!

NB: Equations relating to the simple pendulum, mass-spring system and energy will be dealt with in those sections.

Core equations

a = acceleration *x = displacement* *f = frequency*
t = time *A = amplitude* *T = period*
[and if you use it, ω = angular frequency = 2πf]

NB: If you have never seen the symbol ω in this context - just ignore the equations involving it! Some specifications use it, but many do not.

Acceleration in terms of displacement:
$a = -(2\pi f)^2x$ OR $a = -\omega^2x$

Displacement in terms of time:
$x = A\cos(2\pi ft)$ OR $x = A\cos(\omega t)$
PROVIDED the object starts at a point of maximum displacement

Period in terms of frequency:
$T = \frac{1}{f}$ $\left(\text{and } T = \frac{2\pi}{\omega}\right)$

Additional equations

v = velocity

Displacement in terms of time:
$x = A\sin(2\pi ft)$ OR $x = A\sin(\omega t)$
IF the object starts at the equilibrium position

Maximum velocity:
$v_{max} = 2\pi fA$ OR $v_{max} = \omega A$

Maximum acceleration:
$a_{max} = (2\pi f)^2A$ OR $a_{max} = \omega^2A$

Velocity in terms of displacement:
$v = \pm 2\pi f\sqrt{A^2 - x^2}$ OR $v = \pm\omega\sqrt{A^2 - x^2}$

Acceleration in terms of time:
$a = -(2\pi f)^2A\cos(2\pi ft)$ OR $a = -\omega^2A\cos(\omega t)$
PROVIDED the object starts at a point of maximum displacement

Using the equations

The strategy here is similar to that for using equations of motion at AS - you need to work out which variables you have been given, which variable(s) you need and hence which equation(s) you have to use.

You may also need to use one of these other useful facts:

> ***Acceleration is maximum when displacement is maximum***
> ***Velocity is maximum when displacement (and acceleration) are zero***

Worked Example 1

A toddler in a "baby-bouncer" performs simple harmonic motion with period 4 seconds. The difference in height between the lowest and highest positions of the child is 40cm.

(a) State the amplitude and frequency of the motion
(b) Find the maximum speed of the child

(a) Amplitude = 20cm = 0.2m
(Note the figure given is the difference between the lowest and highest positions - which is double the amplitude)

$$\text{Frequency} = \frac{1}{T} = 0.25\ \text{Hz}$$

(b) $v_{max} = 2\pi fA$
$= 2\pi (0.25)(0.2) = 0.1\pi = 3.14\ ms^{-1}$
(Note units - for this answer to be in metres per second, the amplitude has to be in metres, not centimetres)

Worked Example 2

A particle oscillates with simple harmonic motion of frequency 2Hz. When $t = 0$, the particle is at its position of maximum displacement, and its acceleration has magnitude $3.16ms^{-2}$

(a) Calculate the amplitude of the motion
(b) Write an expression for its displacement at time t
(c) Calculate its speed when it is 0.01m from its equilibrium position.
(d) Find its displacement from equilibrium after 1.1 seconds

(a) We know: $f = 2 \quad a_{max} = 3.16$
(Note: we know $3.16ms^{-2}$ must be the maximum acceleration, since it occurs when the displacement is maximum.)

We want: A
Since we are given a_{max} *it makes sense to use the equation involving this:*
$a_{max} = (2\pi f)^2 A$
f = 2, so:
$3.16 = (4\pi)^2 A$
$A = 3.16/(4\pi)^2 = 0.020\ m.$

(b) Since we know it starts at position of maximum displacement, use
$x = A\cos(2\pi ft)$
$x = 0.02\cos(4\pi t)$

(c) Since we are given x = 0.01, we should use the equation connecting velocity and displacement:
$v = \pm 2\pi f\sqrt{A^2 - x^2}$
$v = 4\pi\sqrt{0.02^2 - 0.01^2}$
$v = 0.22\ ms^{-1}$
We don't need the ± as we are only interested in the size of the speed

(d) We are given time and want displacement, so we use $x = 0.02\cos(4\pi t)$
$x = 0.02\cos(4\pi \times 1.1) = 0.0062m$

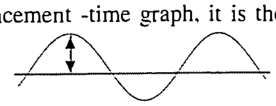
> **Exam Hint** - *You must always put your calculator in* ***radians*** *mode when doing SHM calculations.*

Terminology

The terminology is not usually the direct subject of a question (i.e. you are not often asked "explain what is meant by...") but without knowing what each term means -and using them accurately -it's hard to gain marks.

Displacement *(units - metres)* - This usually refers to the displacement of the object from the fixed centre point. Note that it is displacement, rather than distance, and hence has a sign associated with it (eg negative sign = below the equilibrium point).

Amplitude *(units - metres)* -The size of the maximum displacement - this is always positive. On a displacement -time graph, it is the amplitude of the wave.



Period *(units - seconds)*
The time required for one complete oscillation - the "crest to crest" time.

Frequency *(units - seconds^{-1} or Hz)*
The number of oscillations in one second

Energy exchange in SHM

Questions on energy exchange frequently involve graphs or specific examples of SHM such as the mass-spring system or simple pendulum; these will be dealt with fully in the next two sections. However, all depend on recall and understanding of a few key principles, outlined below.

1. **For a particle performing undamped simple harmonic motion, total energy remains constant.**

2. **(a) When the displacement is <u>maximum</u> (i.e. the particle is at an "end point"), ALL the energy is <u>potential</u>**
(b) When the displacement is <u>minimum</u> (i.e. the particle is at the "centre point"), ALL the energy is <u>kinetic.</u>
(c) Kinetic energy increases as the body approaches the centre point, and decreases as it moves further from it.

You are likely to have to apply these facts in one of the following ways:
- Describing the energy interchange - *this basically involves remembering the above*
- Given a graph showing how one form of energy (eg kinetic) varies, sketch a graph to show how the other form (eg potential) varies. *This involves drawing an "opposite" graph, so that where one has a maximum the other has a minimum etc.*
- Finding the total energy from some information given to you *The question is likely to give you information about either the speed at the centre point, or the height at an extreme point - since there is only one form of energy at these two places, you can deduce the total energy.*

3. **The total energy for a particular case of SHM is proportional to the square of the amplitude of the motion.**

You are likely to be asked to:
- Sketch a graph to show how total energy varies with amplitude
- Work out the total energy for a different system

4. **Potential energy varies as the square of displacement**

You are likely to be asked to:
- Sketch a graph to show how potential energy varies with displacement *- this is just a parabola*
- Sketch a graph to show how kinetic energy varies with displacement *- this is an "upside down" parabola, with the peak where displacement is zero*
- Sketch a graph to show how potential or kinetic energy varies with **time**. *This comes from the expression for displament in terms of time - since potential energy varies as x^2, and $x = A\cos(2\pi ft)$, potential energy must vary as $A^2\cos^2(2\pi ft)$. This is shown in full in the next section.*

Specific oscillating systems

1) Simple pendulum

a) The basic set up
A mass is suspended from an inextensible (i.e. non-stretchy) light string and allowed to hang freely. It is moved slightly sideways, keeping the string taut, and allowed to swing.

This is SHM to a good approximation providing the amplitude of the oscillations is **small** - which means in practice that the angle the mass is displaced through is not more than about ten degrees.

> *Exam Hint - If you have to do any calculations with angles, remember they must be measured in* ***radians****.*

b) Key equation you need to know and use

$$T = 2\pi\sqrt{\frac{l}{g}}$$

T = period
l = length of string
g = acceleration due to gravity (9.81 ms^{-2})

You could also be asked to use formulae for kinetic and potential energy, as well as the other SHM equations.

> ***The period of a simple pendulum does NOT depend on the mass of the bob - just on the length of the pendulum.***

Worked Example
A small piece of lead of mass 40g is attached to the end of a light string of length 50cm and allowed to hang freely.
The lead is displaced to 0.5cm above its rest position, then released.

(a) Calculate the period of the resulting motion, assuming it is simple harmonic
(b) Calculate the maximum speed of the lead mass.

(a) To calculate the period, we use $T = 2\pi\sqrt{\frac{l}{g}}$

So: $T = 2\pi\sqrt{\frac{0.5}{9.81}}$

$T = 1.42s$ *(3sf)*

(b) Here we must use conservation of energy - the potential energy of the mass when it is first displaced is converted to kinetic energy as it passes through its rest position.

Potential energy lost = kinetic energy gained
$mgh = \frac{1}{2}mv^2$
$0.04 \times 9.81 \times 0.005 = \frac{1}{2} \times 0.04 \times v^2$
$v = 0.313ms^{-1}$

2) Mass-spring system

a) The basic set up
A mass is suspended from a vertical elastic spring - it is pulled down or lifted up and then released. This will be SHM provided the spring obeys Hooke's law.
Variations:-
- *a mass is resting on a vertical spring, so the spring starts under compression*
- *a mass is suspended from an elastic string - this will only be SHM provided the string does not go slack*
- *a mass is fixed between two horizontal or two vertical springs*

b) Key equation you need to know and use

$$T = 2\pi\sqrt{\frac{m}{k}}$$

T = period
m = mass of object attached to spring
k = spring constant

Of course, you may also need to use the other SHM equations

Worked Example
A small metal ball of mass 50g is attached to the end of a spring and allowed to hang freely. It is pulled down by 1cm and then released. It performs simple harmonic motion with frequency 20Hz.

(a) Calculate the spring constant
(b) Explain how the the total energy of the ball would be changed if it was pulled down by 2cm instead of by 1cm.
(c) Describe the point at which the kinetic energy of the ball is a maximum

(a) To calculate k, we have to use $T = 2\pi\sqrt{\frac{m}{k}}$

We are given m, but not given T.
But we can work T out by using $T = \frac{1}{f}$

$T = 1/20 = 0.05s$

So $0.05 = 2\pi\sqrt{\frac{0.05}{k}}$

$0.007958 = \sqrt{\frac{0.05}{k}}$

$6.333 = \frac{0.05}{k}$

$k = 790Nm^{-1}$ *(3sf)*

(b) Since total energy is proportional to A^2, if A is doubled, total energy is multiplied by 4

(c) This is the equilibrium point - i.e. the position at which the ball hangs before it is disturbed

Typical Exam Question
A mass of 0.02kg is attached to an inelastic string of length *L*
It is displaced slightly so that it performs simple harmonic motion, with one complete oscillation every two seconds.

(a) Calculate the value of L.
(b) Explain the effect on the period of the motion if a string of 16 times this length were used
(c) An identical mass was connected to a spring and displaced so it performed SHM of frequency 5Hz. Find the spring constant.

(a) T = 2.
So $2 = 2\pi\sqrt{(L/9.81)}$
$(1/\pi)^2 = L/9.81$
$L = 0.994m$

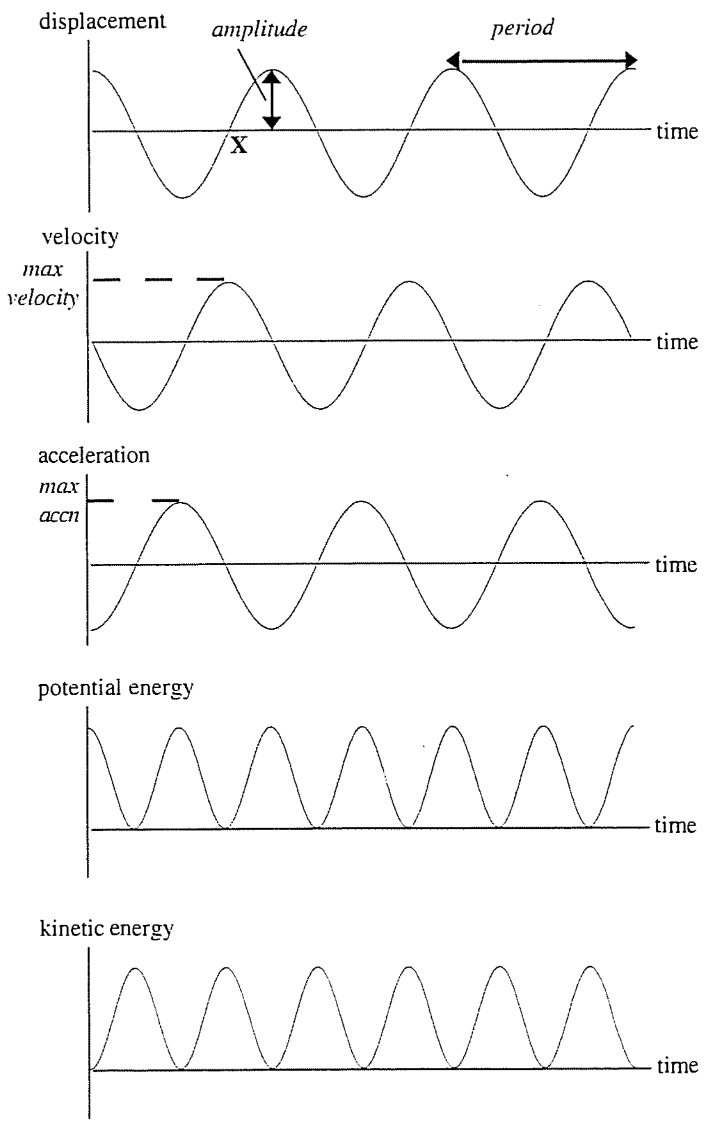
(b) If L is multiplied by 16, $\sqrt{L}$ is multiplied by 4
So the period will increase to 8 seconds

(c) $T = 1/f = 0.2$
So $0.2 = 2\pi\sqrt{(0.02/k)}$
$(0.03183)^2 = 0.02/k$
$k = 19.7\ Nm^{-1}$

Graphs in SHM

This section gives all the standard graphs you are likely to be asked, and includes some key facts about them. Anything you are asked to draw is almost certain to be one of these or a minor variation: the main difference is likely to be the "starting point" (i.e. whether the oscillator starts at an extreme point or a centre point).

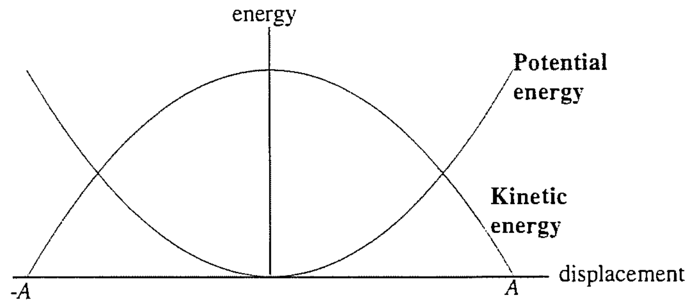
Displacement, velocity, acceleration and energy against time



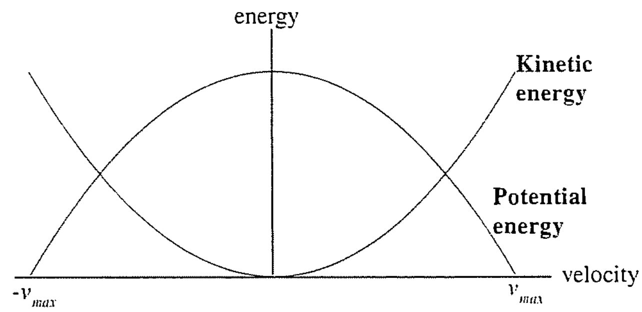
Points to note:
These are all for an oscillator starting at the point of maximum displacement. For one starting at the centre point, start the graphs at the point marked X. For one starting at negative maximum displacement (eg below equilibrium point), start the graph at the first displacement minimum)

- As always, the velocity is the gradient of the displacement-time graph, and the acceleration is the gradient of the velocity-time graph
- Displacement, velocity and acceleration graphs are sinusoidal
- Energy graphs are like $\sin^2 t$ or $\cos^2 t$
- The velocity is 90° out of phase with the displacement
- The acceleration is 90° out of phase with the velocity, and 180° out of phase with the displacement
- The potential energy is maximum when the displacement is maximum or minimum
- The kinetic energy is maximum when the velocity is maximum or minimum

Potential and kinetic energy against displacement



Potential and kinetic energy against velocity



Points to note:
These apply for any SHM

- All the curves are parabolas
- The sum of the two types of energy is always constant
- The kinetic and potential energies swap positions in the graph against velocity (kinetic energy is maximum when velocity is maximimum, which is when displacement is zero)

Total energy against amplitude

Points to note:
This applies for any SHM

- The curve is a parabola
- It also shows how the maximum kinetic energy (or maximum potential energy) varies with amplitude

> ***Exam Hint** - Sketching graphs.*
> *If a question asks you to "sketch" rather than "plot" a graph, it means that you do not have to do it on graph paper, or plot a lot of points in detail. It does NOT mean that you can draw it anyhow! You need to:*
> - *use a ruler for axes*
> - *label axes and curves*
> - *ensure any curves are smooth*
> - *mark on any important values on the axes (eg amplitude, wavelength)*

Practice Questions

1. A mass of 20g is attached to a spring with spring constant 200 Nm^{-1}. The spring is suspended so that the mass hangs freely. The mass is then pulled down a distance A **below** the equilibrium position.

   (a) Draw graphs to show the variation of displacement, velocity and acceleration with time, using the same time scale on each graph.

   (b) Draw a graph to show how the **maximum** kinetic energy of the mass varies with the distance A.

   (c) Find the time after it is released at which the mass first reaches the equilibrium position again.

2. (a) State the conditions necessary for a mass to undergo simple harmonic motion

   A mass performs simple harmonic motion with frequency 0.5Hz. The amplitude of the motion is 2cm.

   (b) Write down

   (i) the maximum speed

   (ii) the maximum acceleration

   of the mass

   A student starts a timer when the displacement of the mass from the centre point of the motion is zero.

   (c) Find the time when the mass is next at this position

   (d) Describe how the potential, kinetic and total energy of the mass have varied during this time.

3. A particle performing SHM has maximum displacement from its centre point 3cm and maximum speed 12π cms^{-1}

   (a) Calculate the period of the motion

   (b) Given that the particle has mass 20g and is oscillating suspended from a spring, find the spring constant.

   (c) A timer is started when the mass is at its highest point. Find the distance it has travelled after 0.3 seconds.

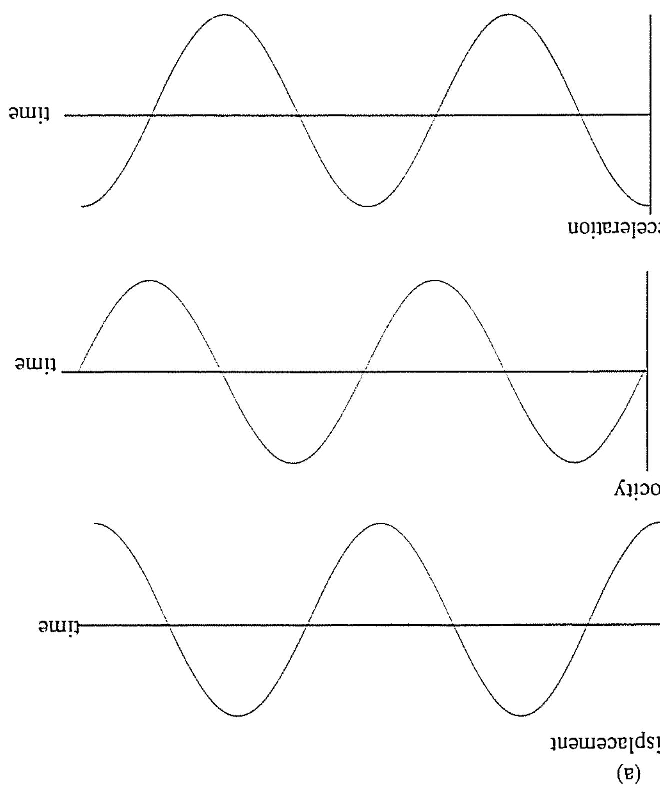
   (d) Calculate the total energy of the particle

   (e) If the amplitude of the motion was reduced to 1.5cm, what would be the new total energy of the particle?

Acknowledgements: *This Factsheet was researched and written by Cath Brown.* ***Curriculum Press, Bank House, 105 King Street, Wellington, Shropshire, TF1 1NU. Chem Factsheets*** *may be copied free of charge by teaching staff or students, provided that their school is a registered subscriber. No part of these Factsheets may be reproduced, stored in a retrieval system, or transmitted, in any other form or by any other means, without the prior permission of the publisher.* ***ISSN 1351-5136***

Answers

1. (a)



(b) See graph on page 4

(c) This is one quarter of the time for an entire oscillation - i.e. one quarter of the period.
Using $T = 2\pi\sqrt{(m/k)}$:
$T = 2\pi\sqrt{(0.02/200)} = 0.0628$s
So time required = 0.0157s

2. (a) See word definition page 1

   (b) (i) $v_{max} = 2\pi fA = 2\pi\,(0.5)(0.02) = 0.0628$ ms^{-1}
   (ii) $a_{max} = (2\pi f)^2A = [2\pi\,(0.5)]^2(0.02) = 0.197$ ms^{-2}

   (c) This is half the time for an entire oscillation (look at the graphs to convince yourself it visits the centre point twice per oscillation)
   $T = 1/f = 2$ seconds
   So time required = 1 second

   (d) Total energy does not change
   Kinetic energy starts at a maximum, declines to zero when mass reaches point of maximum displacement, then increases again to a maximum as mass returns to the centre point.
   Potential energy starts at zero, increases to a maximum when mass reaches point of maximum displacement, then decreases to zero again as mass returns to the centre point.

3. (a) $v_{max} = 2\pi fA$ so $12\pi = 2\pi f(3)$. Hence $f = 2$ Hz
   So period is 0.5s
   Note that since both v and A had length units of cm, it was unnecessary to change into metres

   (b) $T = 2\pi\sqrt{(m/k)}$ so $0.5 = 2\pi\sqrt{(0.02/k)}$
   $k = 3.16Nm^{-1}$

   (c) $x = A\cos(2\pi ft) = 3\cos(4\pi t)$ (*x is in cm*)
   When $t = 0.3$, $x = 3\cos(4\pi\,(0.3)) = -2.43$cm
   (note your calculator must be in radians mode)
   So the mass goes from 3cm above to 2.43 cm below centre point
   So it travels 5.43cm

   (d) Total energy $= \frac{1}{2}mv_{max}^2 = \frac{1}{2}(0.02)(0.12\pi)^2$ (*conversion to m. needed*)
   Total energy = 0.00142J

   (e) Amplitude halved, so total energy divided by 4, giving 0.000355J

Physics Factsheet

www.curriculum-press.co.uk Number 143

Damping in Oscillations

1. Properties of the Damping Force

Many different models of the damping force have been proposed, but you will mostly only have to deal with viscous damping (and the word "viscous" is frequently omitted)

Characteristics of the damping force are:

(a) its magnitude is proportional to the magnitude of the velocity of the oscillating body
(b) it acts in a direction opposite to that of the velocity
(c) it causes kinetic energy to be transformed into other forms (generally heat)

(a) and (b) may be expressed as:

damping force ∝ – (velocity)

or damping force = –c(velocity) where c is a constant
or $F = -cv$

Exam Hint:- *You may be asked to identify properties from a multichoice list, or to provide a list of properties. The three properties given above are the primary properties of a damping force.*

Typical Exam Question: Example 1

Which of the following statements always applies to a damping force acting on a vibrating system:
(a) It is in the same direction as the acceleration
(b) It is in the opposite direction to the velocity
(c) It is in the same direction as the displacement
(d) It is proportional to the displacement

Answer: (b) only

When answering questions on oscillations, it is very helpful to initially identify when they are referring to free vibrations, and when they are referring to forced vibrations, (even though this is not always made clear). So here is a brief outline of the characteristics of each.

2. Free Vibration

Free vibration arises when the system is displaced from its equilibrium position and then released. The only forces acting on the system after release are those generated by the elements of the system i.e. springs, dampers and mass.

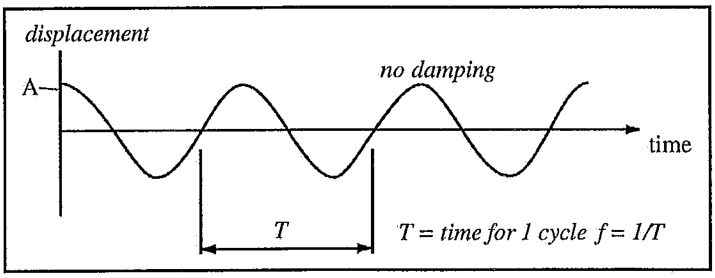
(a) Free Vibration, No damping

If there is no damping, then the system will oscillate with the amplitude of the initial displacement (A) at a frequency (f) determined by the system properties where:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad k = stiffness, \qquad m = mass$$

This is often called the natural frequency.

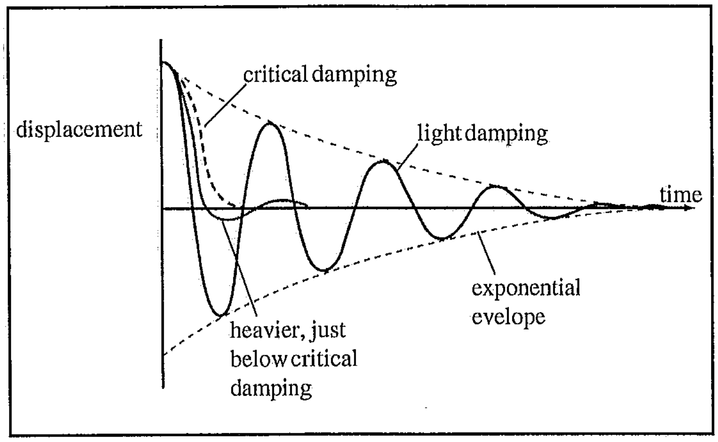
Such oscillation will continue with undiminished amplitude indefinitely.



(b) Free Vibration Damping Exists

If damping exists, and is less than a certain critical value, then the system will oscillate with an exponentially diminishing amplitude and at a slightly reduced frequency (longer periodic time). The frequency is slightly less than the natural frequency given above.

If the damping equals, or is greater than this critical value, then no oscillation occurs. In either case the body will eventually come to rest at its equilibrium position.



Typical Exam Question Example 2

Explain how a free oscillation is affected by the amount of damping.

Answer

Points to make:
(a) Damping causes energy loss from the oscillating system
(b) If there is no damping, oscillation continues undiminished
(c) Less than critical damping successively reduces the amplitude of oscillation in each cycle, until it eventually becomes zero.
(d) Increasing damping reduces the number of cycles needed to reach the stationary condition
(e) If damping is sufficiently large, i.e. critical or above, then there is no oscillation

3. Forced Vibration

Forced vibration arises when the system oscillates due to an input from an external agency, i.e. the world outside the system. This input usually takes the form of an oscillation, or a sequence of pulses.

In this case, the *frequency* of the forced vibration is determined by the "outside agency". (A force applied at a frequency of 50Hz will cause the system to oscillate at 50Hz, even if its natural frequency is different from this.)

The *amplitude* of the vibration is determined by both the vibrating system, and the magnitude and frequency of the input. Resonance may occur when the driving frequency is the same as the natural frequency of the system.

The behaviour is most easily described by plotting the peak amplitude of the oscillation against the applied frequency - a picture known as the "frequency domain" i.e. as in the diagram below.

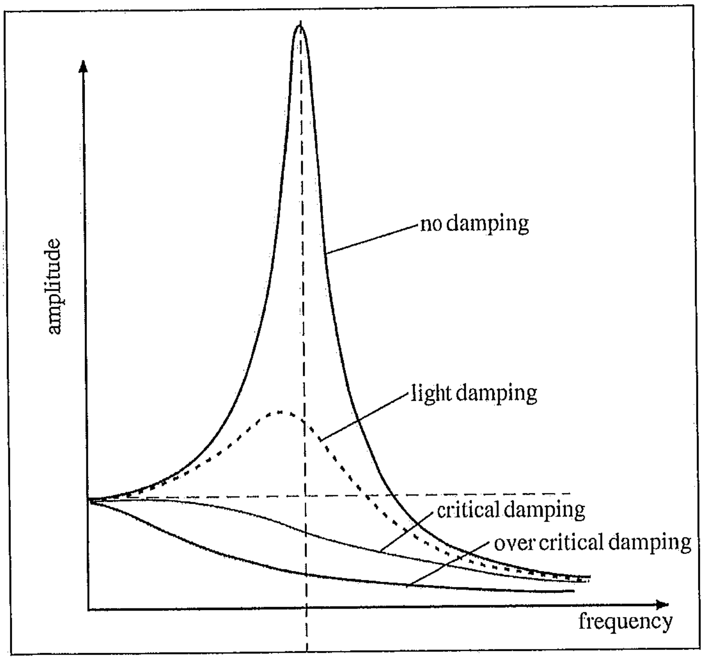
(a) Forced Vibration No Damping
If there is no damping, then the amplitude becomes infinitely great at the so-called "resonant frequency". This is given by:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

which happens to be the same as the natural frequency met in free vibration *(indicated by the broken vertical line on the diagram).*

(b) Forced Vibration Damping Exists
The effect of the introduction of damping is to:
(a) reduce the magnitude of the resonant peak
(b) change the frequency at which it occurs.

Notice that above a certain critical value of damping there is no resonant peak. Also note that the introduction of damping reduces the amplitude of the motion at all frequencies but does not change the frequency – vibrations will still occur at the frequency of the applied oscillations.



Exam Hint
(a) endeavour to understand all the above characteristics of the two regimes, free and forced
(b) examine the question and try to deduce with which regime the question is concerned
(c) endeavour to apply the facts of these characteristics to the situation posed by the question

4. Effect of Damping on the Energy in the System

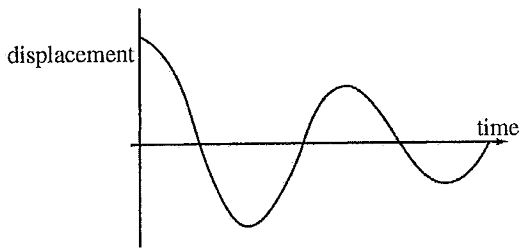
The damping force causes energy to be "lost" by the oscillating system. Damping force is related to the velocity of the moving body (see 1 above). Hence energy "lost" is related to the velocity of the moving body. If there is no damping, then no energy is transferred to heat or sound.

(a) Free Vibration
Energy is supplied only during the initial displacement. This is the total energy. Whilst oscillating, energy varies from potential to kinetic and back.

No damping: *total energy remains the same for all time, none is absorbed.*
Damping exists: *energy is "absorbed" from the total value, which then gradually diminishes, hence reducing the amplitude in successive cycles.*

(b) Forced Vibration
Energy is supplied by the external agency. It makes up for the energy which is lost in each cycle due to the damping force.

Typical Exam Question: Example 3
Given the following displacement-time graph for an oscillating system, show how the damping force varies with time. Also indicate the point in the cycle when the energy "absorbed" is a maximum, and also the point where it is zero.



Answer:
1. Sketch the displacement- time curve as given
2. Add the velocity- time curve, remembering, velocity = gradient of the displacement- time curve
3. Sketch damping force curve, which is proportional to velocity, but acts in the opposite direction.
4. Energy absorbed is related to damping force.
5 Indicate maximum and zero

(Remember that the damping force causes the energy "loss", and the damping force will be greatest when the velocity is greatest. This will occur when the oscillating body is passing through the rest position. And there will be zero energy "loss" when the velocity is zero – this occurs when the displacement is maximum.)

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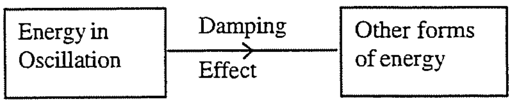
Physics Factsheet

www.curriculum-press.co.uk Number 115

Practical Techniques for Damping Oscillations

Damping is primarily a method of absorbing energy from oscillations for the purpose of controlling or eliminating these oscillations. The kinetic energy of the oscillation can be transferred to kinetic, heat, electrical, magnetic or other forms of energy.

The transfer can be achieved through a variety of means – friction, elastic forces, electromagnetic induction, etc.



Sometimes we are dealing with resonance situations, sometimes with more steady oscillations. In this factsheet, we will look at some example situations where oscillations occur, and how they can be damped.

> ***Exam Hint:*** *Damping always involves the transfer of energy away from the oscillator. Look for the type of energy that is produced in the end (usually heat).*

Resonance:

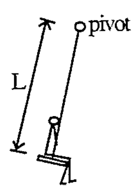
Resonance is a critical type of oscillation where an object oscillates at its natural frequency with greater and greater amplitude, sometimes until it destroys itself. These factors are required for resonance to occur:

(a) An object must have a natural frequency. It will vibrate at this frequency when disturbed e.g. a guitar string being plucked.
(b) A "force" applied regularly at the natural frequency. This could be a mechanical push, or in some cases a varying electrical emf.
(c) There must be a lack of energy loss (damping) from the oscillator. The energy of oscillation must be allowed to keep increasing.

How can resonance be limited? Sometimes by the natural frequency changing e.g. a pendulum only oscillates at a fixed frequency for small oscillations. Or the object might break as the amplitude increases. But we can often control resonance by introducing damping into the system.

> ***Exam Hint:*** *Resonance always involved the transfer of energy from an external source into the oscillator. The amplitude of the oscillation is directly linked to its energy.*

Playground swings



Pushing a child on a swing is an example of mechanical resonance. The swing has a natural frequency depending on its length.

$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$ for small oscillations (measured in Hertz)

If you push at exactly the right time in each oscillation, the amplitude of the oscillations will increase. There is little natural damping.

> **Worked Example 1**
> **(a) Find the natural frequency of a child of mass 20kg on a swing of length 150cm.**
> **(b) Find the period of this oscillation.**
> **(c) Name two sources of damping in this oscillation.**
>
> ***Answer***
>
> (a) $f = \frac{1}{2\pi}\sqrt{\frac{9.8}{1.5}} = 0.41$ Hz The mass has no effect (in theory).
> (b) T = 1/f = 2.4s
> (c) Air resistance and friction at the pivot. Both are relatively small.

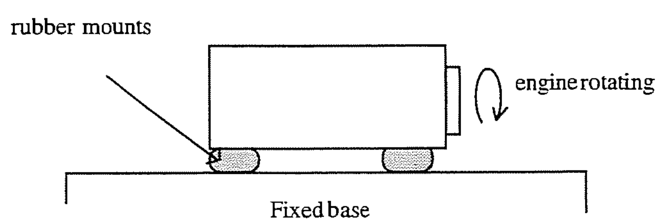
Motors

Many motors or engines rotate at varying speeds while in operation. Examples might be car engines or variable speed electric motors in industry.

Depending on the size, shape, and mass of different parts of the engine, it may have a natural frequency of vibration at a possible operating speed of the engine. This can also happen at multiples of the natural frequency.

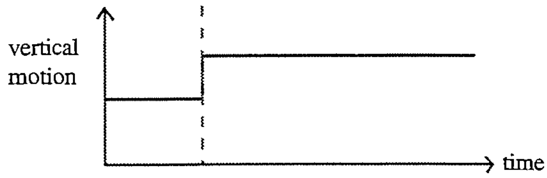
> **Worked Example 2**
> **An engine has a natural frequency of vibration of 1000rpm (revolutions per minute). Find possible angular frequencies of the engine that could lead to resonance.**
>
> ***Answer***
>
> $\omega = 2\pi f = 2\pi \times \frac{1000}{60} = 105 \text{ rad s}^{-1}$
>
> (or multiples e.g. 210, 315, 420, etc)

A standard type of damping here is rubber engine mountings. The mechanical energy of vibration is transferred to heat energy through friction and the hysteresis effect in rubber. The vibrations are limited and the engine doesn't shake itself apart.

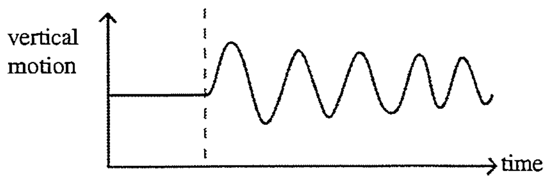


Car Suspension

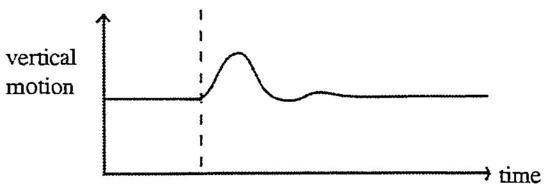
Damping does not always have to be involved with resonance. If a car had no suspension system, there would be a sudden jolt whenever the car hit a bump in the road.



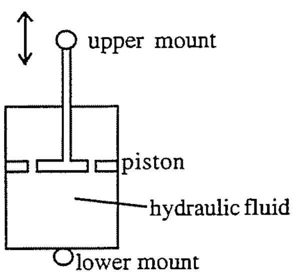
To make the ride safer and more comfortable, springs are introduced into the suspension. The sudden jolt now sets up a more gentle oscillation.



However these extended oscillations themselves are annoying and affect the handling of the car. Shock absorbers in the suspension system damp the oscillatory motion, changing kinetic energy into heat.



The oscillations are quickly eliminated.

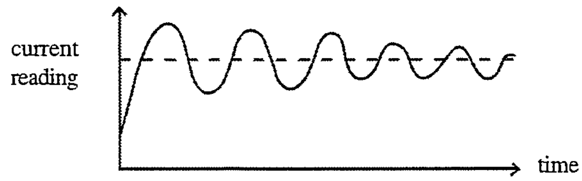
> **How do these shock absorbers work?**
>
> 
>
>
> Simply, as the upper mount moves up and down, hydraulic fluid is forced back and forth through the tiny holes in the piston. There is a great deal of turbulence and friction. The motion is restricted and kinetic energy is transformed into heat.

> **Worked Example 3**
> **Traditionally large American cars have much less damping than sports cars. Can you suggest a reason?**
>
> **Answer:**
> *More damping gives a "harder" ride. However the car holds the road better with fewer oscillations and less leaning on bends.*

Moving coil meters

This is in some ways similar to the situation with the car. Resonance is not involved, but sudden motion causes an unwanted oscillation. The magnetic force (when a current is applied) is opposed by a spring in the meter. The spring sets the needle of the meter oscillating about the correct current reading. Many seconds could pass before the needle settled down to display the reading.

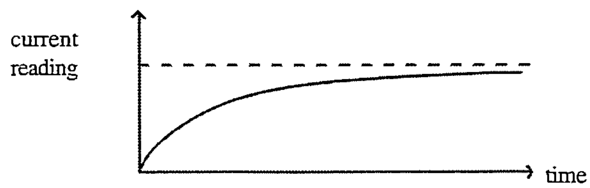
Natural Damping



However we can increase the damping by winding the coil onto a metal former. As the coil moves across the magnetic field, eddy currents are induced in the metal former. Kinetic energy is transferred to electrical energy in the eddy currents. Joule heating causes this energy to end up as heat.

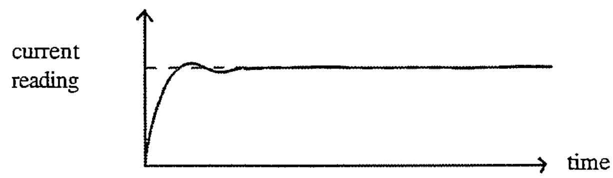
If we transfer kinetic energy too quickly into electrical energy, it takes too long for the meter to reach the correct reading.

Overdamping



If we design everything optimally, then the needle gives the correct reading quickly and accurately.

Critical Damping



The idea is very similar to the car suspension system, but KE is transferred by e.m. induction rather than friction.

> ***Exam Hint:*** *In every case, damping is achieved by changing kinetic energy in the oscillator into some other form of energy. Keep clear in your mind the difference between the means of damping e.g. friction, and the energy transfer e.g. kinetic to heat.*

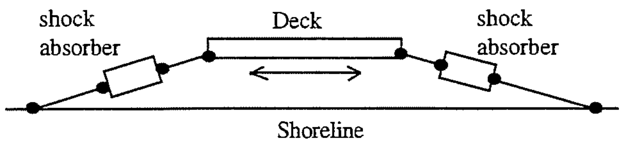
> **Worked Example 4**
> **A galvanometer is overdamped. Suggest a way of redesigning it to decrease the damping.**
>
> ***Answer***
> *Use less metal in the former holding the coil. Or perhaps use metal of a different resistance. There are many ways of altering the strength of the damping.*

The Millennium Bridge

The Millennium Bridge opened in the year 2000 in London. However it swayed so much when crowds walked across it, that it was closed to the public until major strengthening work took place. The culprit was resonance.

The bridge swayed slightly as people walked along it. This was expected. However this caused the pedestrians to step slightly side-to-side in order to keep their balance. As they all did this in time with each other, more and more sideways kinetic energy was transferred to the bridge and the oscillations became bigger and bigger. Resonance was occurring.

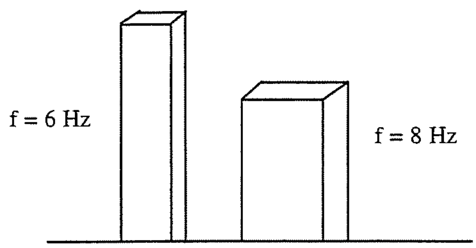
How was this cured?
By a number of techniques including shock absorbers (as used in car suspensions) on the shoreline.



As the bridge oscillates from side-to-side, the shock absorbers transfer kinetic energy to heat energy, reducing the oscillations to an acceptable amplitude.

Earthquake resistant buildings

Worked Example 5
Here are two buildings with different natural frequencies.



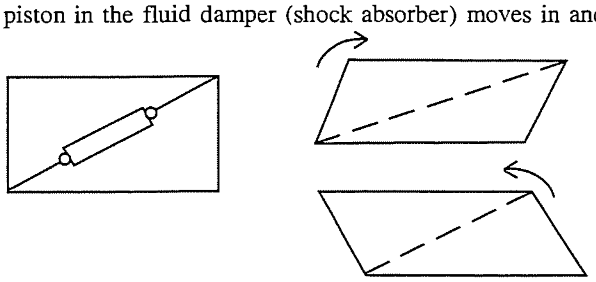
Which one is at risk from earthquakes?

The answer is that both are at risk. Earthquake vibrations arrive with a range of frequencies. Many buildings will be in danger of resonance occurring at their natural frequency.

Obviously buildings in earthquake zones should be built more strongly in general, but there are also techniques used to damp the oscillations. In every case, the dampers are designed to transfer kinetic energy from the oscillations into another form (usually heat). Types of dampers used are:

(a) Metallic dampers: These are metal braces designed to distort, then permanently deform during oscillations. This transfers kinetic energy to heat.
(b) Friction dampers: These have moving parts that slide over each other as the building oscillates. This friction transforms kinetic energy to heat.
(c) Viscous fluid dampers: These again are like shock absorbers in cars. As the building oscillates, kinetic energy is again transformed to heat.

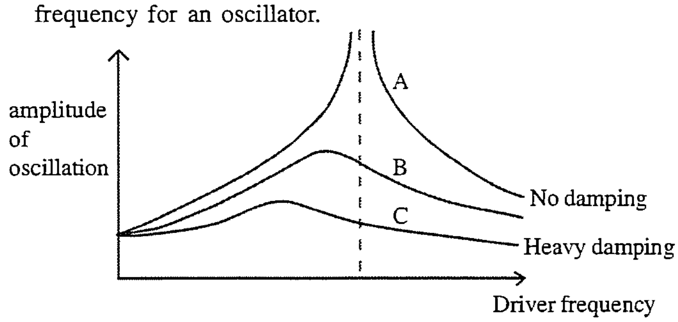
As the building twists clockwise, the diagonal shown increases. Then as the building twists anticlockwise, the diagonal decreases. The piston in the fluid damper (shock absorber) moves in and out.



Exam Hint: *Notice that all three methods result in the kinetic energy being transformed into heat. We spend a lot of time talking about heat as a form of waste energy. Transforming energy into heat can often be a valuable tool e.g. car brakes.*

Practice Questions:

1. A wall clock is driven by a pendulum of length 75cm.
   (a) Find the natural frequency of the pendulum.
   (b) How many times would it oscillate in one day (24hr)?
2. The steering wheel of my car vibrates as the car passes through 20mph.
   (a) What is occurring here?
   (b) How could you tell if the driving force for the oscillations came from the rotation of the engine or the wheels?
   (c) Suggest ways of reducing the problem.
3. Why should tall buildings have a lower natural frequency than shorter buildings?
4. Assume that a damped oscillator loses the same proportion of its energy of oscillation in equal time intervals. If a particular oscillator has 90% of its energy left after 5s, what proportion will be left (a)after 10s? (b)after one minute?
5. Look at this relationship between amplitude and driving frequency for an oscillator.



   (a) What occurs with no damping when the driver frequency matches the oscillator's natural frequency?
   (b) Name three results of increasing the damping.

Answers

1. (a) $f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} = 0.58$ Hz
   (b) $T = \frac{1}{f} = 1.72$s , in one day $\frac{(24 \times 60 \times 60)}{1.72} = 5.02 \times 10^4$ oscillations
2. (a) Resonance.
   (b) Try driving at the same speed in a different gear (different engine speed). If the vibrations don't occur, the engine is the likely source.
   (c) Tighten the fixings for the steering wheel to reduce the oscillations. Clamp rubber or felt in the fixings to absorb energy. Ride a bicycle instead.
3. Think of the buildings as inverted pendulums. A longer pendulum oscillates more slowly. With the buildings the restoring forces would be tension rather than gravity, but the effect would be the same.
4. (a) 0.90 remaining after 5 seconds. So has $0.90^2 = 0.81$ or 81% remaining after 10s.
   (b) After 60 seconds, has $0.90^{12} = 0.28$ or 28% remaining.
5. (a) Resonance
   (b) Broader peak, lower amplitude at all frequencies, frequency of maximum amplitude oscillation decreases.

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Physics Factsheet

September 2002 Number 37

Specific Heat Capacity and Specific Latent Heat

Heat Energy

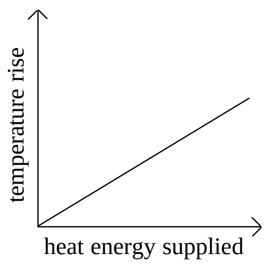
When an object is either supplied with heat energy, or it loses heat energy, there are two possible consequences for the object, a change in its **temperature** and a change of **state** (from liquid to gas for example). This factsheet is concerned with understanding and describing these two consequences of heat energy transfer.

The symbol used for heat energy is *Q*.

> ***The two consequences of a change in heat energy:***
> *Changes in the heat energy of an object can lead to a change in temperature of the object, a change of state of the object or a combination of both of these things.*
> - *If an object is* ***heating up*** *it has had an* ***increase*** *in heat energy and it is said to have had a* ***positive*** *change in heat energy Q will be a* ***positive*** *number.*
> - *If an object is* ***cooling down*** *its heat energy is* ***decreasing*** *and it is said to have had a* ***negative*** *change in heat energy Q will be a* ***negative*** *number.*

Specific Heat Capacity – changes in temperature.

From everyday experience of heating substances, such as heating up water for a cup of coffee, it is easy to believe that the more heat energy that a substance receives, the larger its temperature rise. In fact accurate experiments would show that if you double the amount of heat energy that you give to a material, you would double the increase in temperature of the substance. A graph of temperature rise against energy supplied would be a straight line through the origin, as shown below.



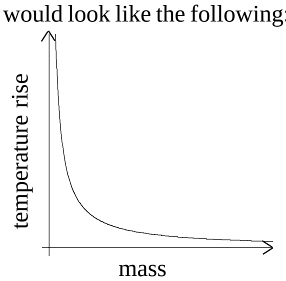
This shows that temperature rise and heat energy supplied are directly proportional to each other.

The symbol for temperature rise is $\Delta\theta$. (The θ symbol represents temperature and the Δ symbol represents a change.) An **increase** in temperature is given a **positive** value.

Mathematically we can express that change in temperature is proportional to the heat energy supplied as follows:

$$\Delta\theta \propto Q$$

Similarly, it is easy to understand that if you take greater and greater masses of a substance and supply each mass with the same amount of heat energy, they will have smaller and smaller increases in temperature. This can be seen through every day experience as well; a kettle may boil a cup full of water in a few minutes, but if the kettle is full of water and you still only switch it on for a few minutes, the water may only be lukewarm. More accurate experiments reveal that if you double the mass of the material being heated, you will only get half the temperature rise if the same amount of heat energy is supplied. A graph of temperature rise against mass of material would look like the following:



This type of relationship shows that temperature rise is inversely proportional to the mass of the material being heated. Mathematically, we can express that temperature rise is inversely proportional to the mass of material being heated as:

$$\Delta\theta \propto \frac{1}{m}$$

> ***The factors affecting the temperature rise of a material.***
> *As the heat energy supplied to a constant mass of material is increased, the temperature rise of the material increases.*
> - *Energy supplied and temperature rise are directly proportional.*
>
> *As the mass of a material increases the temperature rise decreases for a constant amount of energy supplied.*
> - *Mass of material and temperature rise are inversely proportional.*

The two proportionalities given above can be combined into one:

$$\Delta\theta \propto \frac{Q}{m}$$

This combined proportionality can be turned into an equation by inserting a constant of proportionality, *c*.

$$c\Delta\theta = \frac{Q}{m}$$

The constant of proportionality *c* is known as the *specific heat capacity.*

> ***Definition of specific heat capacity***
> *Specific heat capacity is the amount of heat energy needed to increase the temperature of 1 kg of the material by 1 °C.*
>
> $$c = \frac{Q}{m\Delta\theta}$$
>
> *c = specific heat capacity* ($Jkg^{-1}\,{}^oC^{-1}$)
> *Q = heat energy supplied (J)*
> *m = mass of material being heated (kg)*
> *$\Delta\theta$ = increase in temperature (°C)*

Typical exam question
An immersion heater is to heat 50kg of water contained in a copper hot water tank of mass 12 kg.
(a) Calculate how much energy is required to increase the temperature of the water from 20°C to its boiling point.(2)
(b) What assumption have you made in the calculation for part (a)? (1)
Specific heat capacity of water = 4200 Jkg⁻¹ °C⁻¹

(a) Increase in temperature, $\Delta\theta$ = final temperature – initial temperature = 100 – 20 = 80 °C
Heat energy supplied, $Q = mc\Delta\theta$ = (50)(4200)(80)✓ = 16,800,000 J✓
(b) That none of the water has evaporated ✓

Exam Hint: - Don't be concerned when your answers to calculations seem to be large numbers for the amount of heat energy needed for heating up materials such as water. It requires lots of energy to increase the temperature of some materials.

Specific heat capacity and cooling down

The equation for specific heat capacity is not confined to heating up materials, it can also be used when materials cool down. It is important to remember that when a material is cooling down it is losing heat energy so the value for *heat energy supplied, Q,* will be negative. Similarly, the temperature of the material will be falling, and the value for temperature *increase,* $\Delta\theta$, will be negative.

Energy supplied when cooling down

When a material is cooling down its total amount of heat energy is decreasing, and any value for heat energy "supplied", Q, will be negative. This will also mean a negative value for increase in temperature, $\Delta\theta$, as the temperature of the material is decreasing.

Worked Example: An oven tray of mass 120g and made of iron is taken out of the oven and cools by losing 8,000 J of heat energy. When the tray is initially at a temperature of 200°C what will be its final temperature? The specific heat capacity of iron = 470 $Jkg^{-1}\,{}^{o}C^{-1}$

A rearranged specific heat capacity equation can be used to find the increase in temperature for the tray but it is important to include the negative sign in front of the value for heat energy supplied. Also note how the mass of the tray has been converted into kilograms before it has been substituted into the equation.

$\Delta\theta = Q/mc = (-8000)/(0.12)(470) = -142\,{}^{o}C$

This has now given us a negative value for temperature rise, as we would expect because the tray is cooling.

Final temperature = 200 – 142 = 58°C

Specific latent heat – changes in state

If a material is going through a change of state, from solid to liquid, liquid to gas or vice versa, then its temperature will remain constant. The material will still require heat energy to melt it or boil it, but the heat energy is used to alter the arrangement of the molecules of the material. The spacing between the molecules is increased and work is done against the forces of attraction between the molecules. Similarly, the material will also give out heat energy if it is condensing or freezing.

Our existing equation for specific heat capacity **cannot** be used for changes of state.

The more heat energy that is supplied to a material that is changing state, the more of that material will change. Heat energy supplied, *Q*, and mass of material changing state, *m*, are directly proportional.

$Q \propto m$

Once again, this proportionality can be changed into an equation by inserting a constant of proportionality, *l*.

$Q = lm$

Q = Heat energy supplied (J)

m = Mass of material changing state (kg)

The constant of proportionality, *l*, is called the specific latent heat and it has the units Jkg^{-1}.

Every material has two values of specific latent heat, one value for a change of state from a solid to a liquid, called the specific latent heat of fusion ("fusion" is another word for "melting"), and one value for a change of state from a liquid into a gas, called the specific latent heat of vaporisation.

The symbol used for both specific latent heat of fusion and specific latent heat of vaporisation is *l*.

Definition of specific latent heat

Specific latent heat of fusion is defined as the amount of energy required to turn 1 kg of a solid into a liquid at a constant temperature.

Specific latent heat of vaporisation is defined as the amount of energy required to turn 1 kg of a liquid into a gas at a constant temperature.

$l = Q/m$

l = specific latent heat (Jkg^{-1})

Q = heat energy supplied (J)

m = mass of material changing state (kg)

Gaining and losing heat energy when changing state

- *If a substance is changing from a solid to a liquid or from a liquid to a gas then heat energy is being supplied to a substance and Q is positive.*
- *If a substance is changing from a gas to a liquid or a liquid to a solid then it is losing heat energy and the heat energy "supplied", Q, is negative.*

Typical exam question

A kettle is filled with 2.0kg of water.

(a) Calculate how much energy is required to increase the temperature of the water from 20°C to boiling point, 100°C. (3)

(b) If the kettle is left running so that it supplies another 12000J of energy, how much of the water will have evaporated and turned from a liquid into a gas.(2)

specific heat capacity of water = 4200 $Jkg^{-1}\,{}^{o}C^{-1}$

specific latent heat of vaporisation of water = 2.3×10^{6} Jkg^{-1}

(a) Change in temperature, $\Delta\theta$ = final temperature – initial temperature = 100 – 20 = 80 °C ✓

Heat energy supplied, $Q = mc\Delta\theta$ = (2)(4200)(80) ✓ = 672,000 J ✓

(b) Mass evaporated, $m = Q/l = (12000)/(2.3\times10^{6})$ ✓ = 0.0052kg = 5.2g ✓

Qualitative (Concept Test)

(1) What are the two possible consequences of heating or cooling a sample of material?

(2) (a) If different masses of the same material are supplied with equal amounts of heat energy, how will the temperature increase depend on the mass?

(b) Sketch a graph of temperature increase against mass.

(3) Define specific heat capacity as an equation and in words.

(4) What happens to the temperature of a substance as it changes from a liquid to a solid?

(5) Define, in words, the specific latent heat of vaporisation.

(6) What is the difference between latent heat of vaporisation and latent heat of fusion?

Quantitative (Calculation Test)

(1) What is the specific heat capacity of aluminium if it takes 18,200 J of energy to increase the temperature of a 5.0 kg block of aluminium by 4.0 °C?(2)

(2) How much heat energy will be supplied to 1.5kg of water when it is heated from 20°C to 80°C? (2)

(Specific heat capacity of water = 4,200 $Jkg^{-1}\,{}^{o}C^{-1}$)

(3) How much heat energy will be given out by a block of copper with a mass of 2.0 kg, which is cooling down from 150 °C to 20 °C? Specific heat of copper = 390 $Jkg^{-1}\,{}^{o}C^{-1}$ (2)

(4) How much heat energy will have to be supplied to melt 0.50kg of ice? Specific latent heat of fusion of ice = 330,000 J/kg. (2)

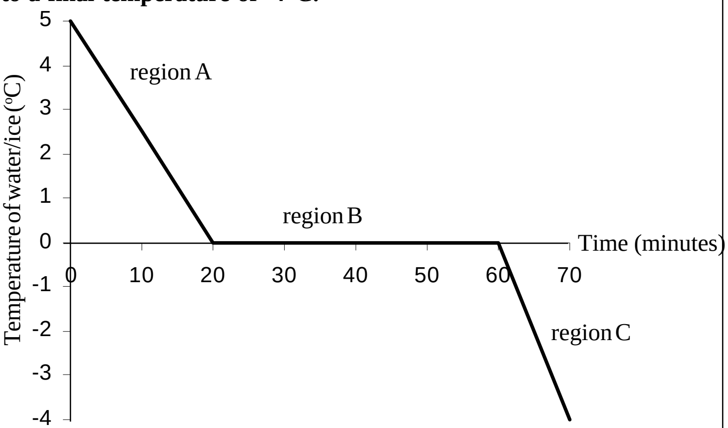
(5) (a) What is the total amount of energy needed to heat up a copper kettle of mass 1.5 kg, from 20°C to boiling. The kettle is filled with water of mass 2.5 kg.(5)

(b) Give 2 assumptions made in your calculation.(2)

Exam Workshop

This is a typical answer that might might be given by a weak student in an examination. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

The graph below shows how the temperature is changing over a period of time for 25g of water that is being cooled, turned into ice, and taken to a final temperature of -4 °C.



For this question you will need to know:

specific heat capacity of water = 4200 $Jkg^{-1}\,{}^{o}C^{-1}$;

specific latent heat of fusion of water/ice = 3.3×10^5 Jkg^{-1}.

(a) In which region of the graph, A, B, or C, is the water changing into ice? (1)

C ✗ *0/1*

In region C the water has completely turned to ice and has started to cool further. The temperature of all materials remains constant as they change state. The student should be looking for the region of the graph where the temperature does not change.

(b) How much heat is extracted from the water to cool it during the first 20 minutes of the graph? (2)

$Q = mc\Delta\theta = (25)(4200)(5)$ ✗ $= 525{,}000J$ ✗ *0/2*

The student has used the correct equation for specific heat capacity but has not converted the value for mass into kilograms and the value for time must also be converted into seconds.

(c) How much heat energy is extracted from the water during the time that the temperature remains constant at 0 °C? (2)

$Q = ml = (25)(3.3 \times 10^5)$ ✓ $= 8{,}250{,}000J$ ✓ ecf *2/2*

The student has correctly used the equation but with the same mistake in the units of mass. This mistake would probably not be penalised a second time in the same question.

(d) If a total of 210J of energy are extracted from the ice in the final 10 minutes shown by the graph, calculate the specific heat capacity of ice. (2)

$c = Q/m\Delta\theta = (210)/(25)(-4) = -0.21\ Jkg^{-1}\,{}^{o}C^{-1}$ ✗ *0/2*

The student has not given the heat removed a negative sign (as energy has been extracted from the ice). This has given an incorrect negative answer, which is also a thousand times too small because of the recurring mistake of quoting mass in g instead of kg.

Examiner's Answers

(a) *C*

(b) $Q = mc\Delta\theta = (0.025)(4200)(5) = 525\ J$

(c) $Q = ml = (0.025)(3.3 \times 10^5) = 8{,}250\ J$

(d) $c = Q/m\Delta\theta = (-210)/(0.025)(-4) = 2{,}100\ Jkg^{-1}\,{}^{o}C^{-1}$

Quantitative Test Answers

(1) $c = Q/m\Delta\theta = (18{,}200)/(5)(4) = 910\ Jkg^{-1}\,{}^{o}C^{-1}$.

(2) $Q = mc\Delta\theta = (1.5)(4200)(80-20) = 378{,}000J$

(3) $Q = mc\Delta\theta = (2)(390)(20-150) = -101{,}400J$. *Note the answer is negative, showing the copper block to be cooling.*

(4) $Q = ml = (0.5)(-330{,}000) = -165{,}000J$. *Note the answer is negative showing that the ice is giving out energy and not being supplied with it.*

(5) *(a) Energy needed to heat up water* $= mc\Delta\theta = (1.5)(4200)(100-20) = 504{,}000\ J$

Energy needed to heat up kettle $= mc\Delta\theta = (2.5)(390)(100-20) = 78{,}000\ J$

Total energy needed $= 504{,}000 + 78{,}000 = 582{,}000\ J$

(b) *The copper heats up to the same temperature as the water. There is no energy loss to the surroundings of the kettle such as air and the surface that it stands on.*

Acknowledgements: *This Physics Factsheet was researched and written by* ***Jason Slack.***

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Physics Factsheet

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Molecules, Moles and Mass

This Factsheet focuses on questions and calculations about gases. This includes the Ideal gas equation and kinetic theory and covers molecules, moles, kinetic energy of a gas particle and rms velocity. There are several common mistakes which will be explained.

Ideal Gas Equation and the Mole

The equation of state for an ideal gas is given by pV=nRT, otherwise known as the Ideal Gas Equation. The pressure, p, the volume, V and the temperature, T are obvious from their symbols. The Molar Gas Constant, R is 8.31J/K/mol. The number of moles of gas, n, needs more explanation.

When you breathe in deeply, your lungs contain around 6 litres of air (depending on your size and other factors). These 6 litres of air consists of approximately 1.2×10^{24} individual particles: mostly molecules of nitrogen, oxygen and some carbon dioxide. A mole represents 6×10^{23} individual particles: either atoms or molecules.

This means our lungs contain around 2 moles of air, a much more down-to-earth figure. The exact number is based on carbon-12. If you have exactly 0.012kg of C-12, you have 1 mole of particles, which is actually 6.022×10^{23} particles and known as Avogadro's constant (L or N_A). One single atom of C-12 has a mass of 12u and one mole of C-12 atoms has a mass of 0.012kg.

> *One mole is 6.022×10^{23} particles. This number of particles, atoms or molecules, is known as Avogadro's constant, N_A.*

We can also consider molecules as well as atoms. This is more useful for gas calculations as gases are more commonly molecular and not atomic. One molecule of O_2 has a mass of 32u. Therefore 1 mole of O_2 has a mass of 0.032kg.

Particle	Particle mass (u)	Mass of 1 mole
Carbon atom	12	0.012kg
Oxygen atom	16	0.016kg
Helium atom	4	0.004kg
O_2 molecule	32	0.032kg
CO_2 molecule	44	0.044kg
N_2 molecule	28	0.028kg

> ***Exam Hint:*** *In Chemistry, molar masses are given in grams. For Physics calculations, remember to keep all masses in kilograms to avoid mistakes.*

Example 1

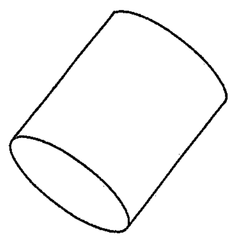
1. ***What is the atomic mass of Argon (in atomic mass units, not kg)?***
2. ***What is the mass of 1 mole of Argon gas?***
3. ***What is the mass of 4.5 moles of sulphur dioxide gas?***
4. ***How many moles are there in 0.038kg of carbon monoxide molecules?***

Answers

1. *The atomic mass of Argon is 40 u (39.95).*
2. *One mole of Argon has a mass of 0.04kg.*
3. *The mass of a single SO_2 molecule = 32 + (2 x 16) = 64u*
   1 mole of SO_2 has a mass of 0.064kg
   4.5 moles of SO_2 has a mass of 0.064kg × 4.5 moles = 0.288kg
4. *The mass of a single CO molecule = 12 + 16 = 28u*
   1 mole of CO has a mass of 0.028kg
   0.038kg / 0.028kg = 1.36 moles.

Ideal Gas Equation calculations

A sealed container holds carbon dioxide gas. The pressure inside is 10 atmospheres, the volume of the container is 200cm³ and the temperature is 30°C. How many moles of gas are there in the container?



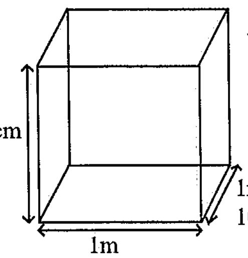
V = 200 cm³
T = 30°C
P = 10 atmospheres

There are three common mistakes that could be made when answering this question:

Pressure must be in units of N/m² or Pascals. Volume must be in units of m³ and the temperature must be in Kelvin on the absolute temperature scale.

Converting the pressure first: One atmosphere is equivalent to 1.013×10^5 Nm^{-2} or Pa. So $10 \times 1.013 \times 10^5 = 1.013 \times 10^6$ Pa

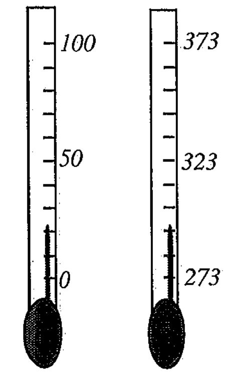
The volume must be in m³. How do we convert from cm³ to m³? Picture a box, with sides 1m by 1m by 1m. Each side is 100cm long. To find out how many cm³ there are in the box, multiply 100cm by 100cm by 100cm= 1×10^6 cm³ or 1 million cm³. Volumes are often given in litres. 1 litre contains 1000cm³, so there are 1000 litres in 1m³.



So $200\text{cm}^3/1 \times 10^6\text{cm}^3 = 2 \times 10^{-4}$ m³

Exam Hint: It is easy to make a mistake when converting from cm^3 to m^3. If you have $200cm^3$, the result must be a SMALL fraction of a cubic metre.

Exam Hint: When using the Ideal Gas Equation the temperature must be in Kelvin , according to the absolute temperature scale. 0K is equivalent to -273.15°C. 0°C is equivalent to 273.15K. Add 273.15 to any temperature in °C to change it to Kelvin.



Celsius (°C) temperature scale *Kelvin (K) temperature scale*

So 30°C + 273.15 = 303.15K

pV=nRT

becomes $\frac{pV}{RT} = n$ *after rearranging*

$$\frac{1.013 \times 10^6\ Pa \times 2 \times 10^{-4} m^3}{8.31 \times 303.15K}$$

= *0.080 moles.*

Example 2
1. ***How many molecules were in this sealed container?***
2. ***What is the mass of the gas inside this sealed container?***

Answers
1. *0.080 moles × 6.022 × 10^{23} = 4.82 × 10^{22} molecules.*
2. *One molecule of carbon dioxide has a mass of :*
   12 + 2 × 16 = 44u. So one mole is 0.044kg
   0.080 moles × 0.044kg = 3.52 × 10^{-3}kg

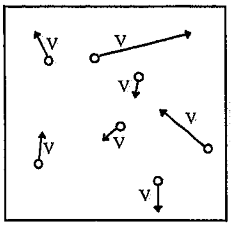
Kinetic Theory
Using the kinetic theory, we can derive a very useful equation:
$pV = 1/3Nm\overline{v^2}$

This equation links large scale easy-to-measure quantities like pressure, p and volume, V to small scale quantities like the mass of a particle, m (in kg), number of individual particles, N, and average squared speed, $\overline{v^2}$.

Exam Hint: The number of moles, n, and the number of individual particles, N, are VERY easy to confuse. One way to help remember the difference: n (lower case) will be a relatively small number, N (capital letter) will be a very large number.

Average squared speed is one concept that people sometimes struggle with. The N particles in any gas will be travelling with a range of speeds and every particle has a different velocity, v. This means that the squared speed for each particle, v^2, is also different. For calculations involving many particles, we use an average of this squared speed: $\overline{v^2}$, units m^2s^{-2}. (The reason we are more concerned with squared speed, rather than speed itself, is because squared speed is needed for kinetic energy calculations.)

*$\overline{v^2}$ is the **average of the squared speeds** of all of the particles in the gas. This is often known as the **mean squared speed.** Some textbooks write this as $\overline{c^2}$.*


Gas particles travel with a range of velocities

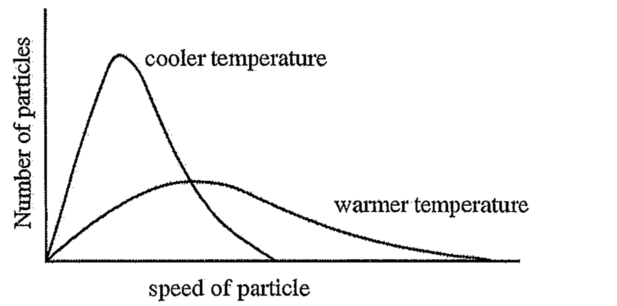
So, on average, how fast are the particles travelling?

*Take the square root of the mean squared speed, $\sqrt{\overline{v^2}}$, this gives a good **indication** of how fast the particles travel on average.*

$\sqrt{\overline{v^2}}$ is known as the root-mean-squared-velocity (or rms velocity).

A small proportion of the particles will be travelling very slowly; another small proportion will be travelling very fast. The majority of the particles will be travelling between these two extremes. If you look at the distribution of the particles across the different speeds, it forms a broad hump. This is known as the Maxwell-Boltzmann distribution. At higher temperatures, the particles are distributed across a wider range of speeds.

The Maxwell-Boltzman Distribution



Kinetic energy
The kinetic theory provides the relationship for the average kinetic energy of a **single** particle and temperature of the gas:

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT$$

The left hand side of the equation is the kinetic energy of a single particle. The Boltzmann constant, k is $1.38 \times 10^{-23} JK^{-1}$ and the temperature, T, must be on the absolute (Kelvin) scale. This simple equation gives a whole new explanation for temperature.

The temperature of the gas is directly dependent on the kinetic energy of the individual particles. This equation tells us that, for every 1K increase in temperature, the average kinetic energy of the particle increases by roughly 2×10^{-23}J. This is true for any gas particle: atom or molecule. If you mix Helium atoms with Carbon dioxide molecules, each of the two different particles will have the same average kinetic energy. Of course, the masses are quite different, so one must be travelling much faster on average.

> ***Exam Hint:*** *The average kinetic energy for each particle in a mixture of gases at the same temperature is the SAME. Hydrogen molecules will have the same average kinetic energy as oxygen molecules when mixed. As their masses are different, they will have different mean-squared-velocities.*

> *Example 3*
> 1. ***Calculate the rms velocity of a Helium atom at room temperature.***
> 2. ***Which molecules will have more kinetic energy in a mixture of nitrogen and hydrogen molecules at the same temperature?***
> 3. ***On average, will the hydrogen or nitrogen molecules be travelling fastest?***
>
> ***Answers***
>
> 1. *Rearrange* $\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT$ *to give* $\overline{v^2} = \frac{3kT}{m}$
> $= \frac{3 \times 1.38\times10^{-23}JK^{-1} \times 293K}{4 \times 1.661\times10^{-27}\,kg}$
>
>   *This gives* $\overline{v^2} = 1.83\times10^6\ m^2s^{-2}$.
>
>   *So* $\sqrt{\overline{v^2}} = 1.35\times10^3 ms^{-1}$ *or 1.4km/s.*
> 2. *As kinetic energy of a single molecule/ atom is given by 3/2kT, the nitrogen and hydrogen molecules will have the same average kinetic energy.*
> 3. *The mass of the nitrogen molecules is fourteen times greater than the hydrogen molecules, so* $\overline{v^2}$ *must be fourteen times greater for the hydrogen molecules.* $\sqrt{\overline{v^2}}$ *for hydrogen molecules is 3.8x greater than for the nitrogen molecules.*

Total Internal energy

Most of the energy within a gas is due to the particles moving about. In a molecular gas (like CO_2), the particles can have energy through spinning around or vibrating about their chemical bonds. However, most of the energy is kinetic energy due to the movement of the particles, which we call translational kinetic energy.

To calculate the total internal energy of a gas $U = 1/2Nm\overline{v^2}$. This is simply the product of the number of molecules, N and the average kinetic energy of a single particle, $1/2m\overline{v^2}$.

> *Example 4*
> 1. ***What is the total internal energy of 4 moles of oxygen molecules at 25°C?***
> 2. ***How many particles are there in a gas with 25,000 Joules total internal energy at 600°C?***
>
> **Answers**
>
> 1. $U = 1/2Nm\overline{v^2}$ *and* $1/2m\overline{v^2} = 3/2kT$ *so* $U = 3/2NkT$
> $T = 25+273.15K = 298.15K$
> $N = 4 \times 6.022x10^{23} = 2.4\times10^{24}$ *particles*
> $U = 3/2 \times 2.4x1024 \times 1.38x10^{-23} \times 298.15K = 14.8kJ$
> 2. $U = 3/2NkT$ *becomes* $2U/3kT = N$
> $N = \frac{2 \times 2.5\times10^4 J}{(3 \times 1.38\times10^{-23} \times 873.15K)} = 1.38 \times10^{24}$ *particles*

Practice Questions

1. Calculate the rms velocity of hydrogen molecules, oxygen molecules, argon atoms and radon atoms at room temperature.
2. Compare the average kinetic energy and rms velocity of helium atoms and carbon dioxide molecules in a mixture of gases.
3. A cylinder contains gas at 120 atmospheres at 20°C. A warning sign states that the maximum safe pressure is 240 atmospheres. What is the maximum safe temperature of the gas?
4. What is the average kinetic energy of a particle at 100°C?
5. What is the total internal energy of 1.2 moles of steam at 177°C?
6. The pressure inside a blow-up balloon is 1.3 atmospheres. The balloon is roughly 20cm across. Calculate the volume of the balloon by assuming it is spherical. How many moles of particles will be in the balloon at room temperature?
7. How many moles are there in 10.2kg of uranium hexafluoride gas? Uranium hexafluoride is a molecular gas, each molecule consists of one uranium atom and six fluorine atoms. Use the most common isotope for each element in your calculation.

Answers

1. 1.9km/s, 480m/s, 430m/s, 180m/s
2. Average kinetic energy of helium atoms and carbon dioxide molecules is the same. $\sqrt{\overline{v^2}}$ for helium is 3.3 times greater than for carbon dioxide molecules.
3. 586K
4. 7.7x10-21J
5. 6.7kJ
6. 0.23 moles
7. 29.0 moles

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Physics Factsheet

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The Gas Laws

This Factsheet will explain:
- **how Boyle's Law is arrived at experimentally;**
- **how Charles' Law is arrived at experimentally;**
- **the Kelvin temperature scale;**
- **how the Pressure Law is arrived at experimentally;**
- **how these three are combined into the Universal Gas Law;**
- **how to use the Gas Laws to do simple calculations.**

Before studying this Factsheet, you should have an idea of how particles are arranged in solids, liquids and gases and know that because the particles in a gas are far apart they can be pushed together easily. You should know that gases exert a pressure on the walls of their container by the particles hitting the walls. You should know that pressure = force/area. You should also be familiar with the Celsius scale of temperature and the concept of a mole of a substance (you find the moles of a substance by dividing its mass in grammes by its molecular mass).

Boyle's law

From your knowledge of how particles are arranged in a gas it should seem likely that if you decrease the volume of a gas, its pressure will increase, because the particles will have less far to travel between hitting the walls and so will hit the walls more often. It was Robert Boyle who investigated this effect more carefully using a thick-walled pressure vessel with variable volume, which could be altered by turning a screw. The pressure was measured using a manometer. He arrived at the result that the pressure of a fixed mass of gas is inversely proportional to its volume. i.e. if you halve the volume the pressure will double, provided that the temperature does not change. This can also be expressed as:

$$pV = \text{constant}$$

> ***Boyle's Law:***
> *For a fixed mass of gas at constant temperarure PV = constant*

Typical Exam Question

1. **5*l* of a gas is at a pressure of 1 atmosphere. What will its volume be if the pressure is increased to 1.5 atmosphere, with no change in temperature?**

   pV = constant, so $5 \times 1 = V \times 1.5$

   So $V = \frac{5}{1.5} = 3.33l$

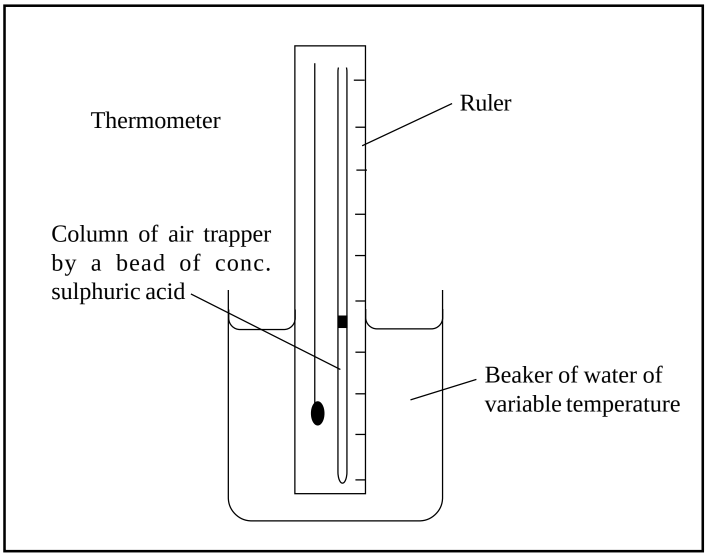
2. **A gas occupies a volume of 3*l* at a pressure of 1.9 atmosphere. What pressure would reduce its volume to 1.6*l* assuming there is no change in temperature?**

   pV = constant, so $3 \times 1.9 = P \times 1.6$

   So $p = \frac{3 \times 1.9}{1.6} = 3.56$ *atmos*

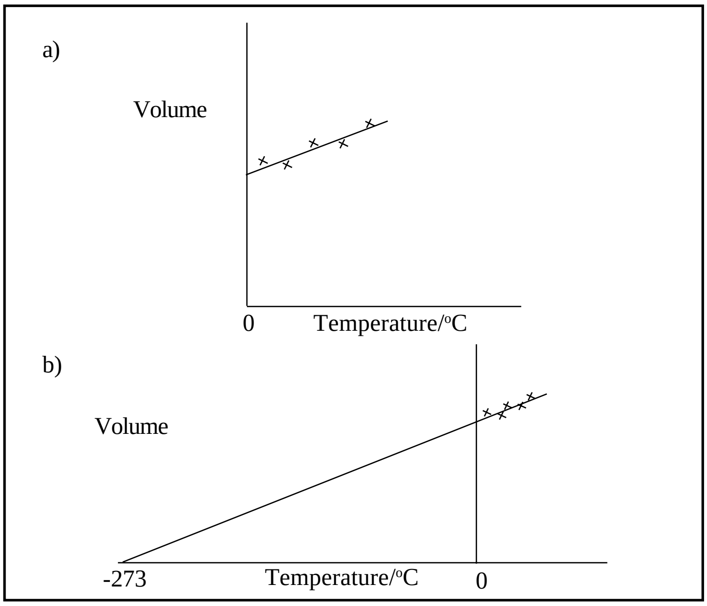
Charles' law

Fig. 1 Apparatus for investigating the volume of a fixed mass of gas at different temperatures



The apparatus shown in fig.1 can be used to find the volume of a fixed mass of gas at different temperatures. When the results are plotted on a graph of temperature, in ^{0}C, against volume, as in Fig.2 a) the result is a straight line **but it does not go through the origin**. When the graph is replotted to show the intercept on the temperature axis, as in Fig.2 b) it is found that the intercept is at – 273.16 ^{O}C. This temperature is known as ***absolute zero*** and it represents the temperature at which the volume of the gas would be zero.

Fig. 2 Volume against temperature for a fixed mass of gas at constant pressure



It is very important to realize that in extrapolating the graph to absolute zero you are making the assumption that the gas continues to behave in exactly the same manner. A gas that does this is called an ***ideal gas***. In practice, no gases behave exactly as ideal gases, but many approximate to ideal gas behaviour, so the concept is useful.

The Kelvin Temperature Scale

If we shift the origin of the graph to absolute zero, then the graph of volume against temperature becomes a straight line through the origin. This means that we can say that volume is proportional to the **new** temperature. This new temperature scale is called the ***kelvin temperature*** scale (also ***the absolute temperature*** scale) and is arrived at by adding 273 to the temperature in ^oC

Practice Question

1. **Give the following Celsius temperatures in kelvin**
   a) 23 ^{0}C   b) 94 ^{0}C   c) 120 ^{0}C

   a) $23\,^0C = 23 + 273 = 296K$
   b) $94\,^0C = 94 + 273 = 367K$
   c) $120\,^0C = 120 + 273 = 393K$

2. **Give the following kelvin temperatures in ^{0}C**
   a) 420K   b) 368K   c) 420K

   a) 420K = 420 – 273 = 147 ^{0}C
   b) 368K = 368 – 273 = 95 ^{0}C
   c) 315K = 315 – 273 = 42 ^{0}C

Provided the temperatures are on the kelvin scale, we can express Charles' law as:

$$V \propto T$$

Charles' Law:
For a fixed mass of gas at constant pressure
$V \propto T$ *where* *V is the volume*
T is the temperature (Kelvin (K))

***Exam Hint:* Always** *remember to use kelvin temperatures for any gas calculations. The symbol T is used for a temperature in Kelvin.*

Typical Exam Question
The temperature of 1 litre of gas is raised from 10 ^oC to 20 ^oC. What will be its volume at the higher temperature, if the pressure is kept constant?

$V \propto T$, so $\frac{V}{T}$ is constant.

$$\frac{1}{283} = \frac{V}{293}$$

$$V = \frac{293}{283} = 1.04l$$

The Pressure law

So far we have considered:
- a fixed mass of gas at constant temperature and found a relationship for *p* and *V*
- a fixed mass of gas at constant pressure and found a relationship for *V* and *T*

Not surprisingly, there is also a relationship between *p* and temperature for a fixed mass of gas at constant volume. This is known as the Pressure law.

Plotting *p* against temperature in ^oC for a fixed mass of gas results in an intercept on the temperature axis of –273.16, just as for *V* against temperature. So using temperature on the kelvin scale applies to the Pressure law as well.

The Pressure law:
For a fixed mass of gas at a constant volume: $\frac{p}{T} = constant$

Typical Exam Question
The temperature of a fixed volume of gas at a pressure of 2.0×10^5 Pa is increased from 23 ^{0}C to 37 ^{0}C. What will be the new pressure?

$\frac{p}{T} = constant$, so

$$\frac{2.0 \times 10^5}{296} = \frac{p}{310}$$

$$p = \frac{2.0 \times 10^5 \times 310}{296} = 2.1 \times 10^5\,Pa$$

Exam Hint: *You may find volumes quoted in litres or m^3, pressures quoted in atmospheres or pascals (Pa) – 1 Pa = 1Nm^{-2}. If you are using Boyle's or the Pressure Law, just leave it in whatever unit it is given in.*

The Ideal Gas Equation.

The three gas laws can be combined into a single equation, which can cope with a change in more than one variable at once. This combination is known as **the ideal gas equation.** It can be stated as:

$\frac{pV}{T}$ = constant for 1mole of gas

The constant for 1 mole of gas is known as the molar gas constant, *R* and has the value 8.3 JK^{-1}mole^{-1}

If there is more than 1 mole of gas then we need to multiply by the number of moles.

The final statement of all this information is: $pV = nRT$, where *n* is the number of moles.

The ideal gas equation:

$pV = nRT$ *Where* p = *pressure in Pa ,*
V = *volume in* m^3
n = *the number of moles of gas*
R = *the molar gas constant*
T = *the temperature in kelvin.*

Exam Hint*: In using the ideal gas equation, it* <u>*does*</u> *matter which units you use. When the pressure is in Pa the volume must be in m^3 and the temperature in K otherwise R would not be 8.3 JK^{-1}mol^{-1}.*

Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's mark scheme is given below.

a) State and explain the *ideal gas equation*. **[2]**

pV = nRT ✓✗ 1/2

> Although correct, this is insufficient for both marks. The candidate should have explained the meaning of the terms and stated that this is for an ideal gas.

b) Explain why it is necessary, when using the gas laws, to express the temperature on the kelvin temperature scale. **[2]**

Because it won't work otherwise. ✗✗ 0/2

> This has not answered the question. The examiner is looking for an understanding that for proportionality, the graph must be a straight line, through the origin, so the origin must be shifted to where the graph cuts the temperature-axis.

c) The volume of a balloon containing gas is 0.4*l* at a temperature of 15°C, and a pressure of 101kPa. What will be the new volume of the balloon, if the temperature rises to 25° C and the pressure decreases to 95kPa? **[3]**

$\frac{pV}{T}$ = constant

so $\frac{101 \times 0.4}{15} = \frac{95 \times V}{25}$

$V = \frac{101 \times 0.4 \times 25}{15 \times 95}$ = 6396.7l ✓✗✗ 1/3

> The candidate is using the correct method but, despite the gentle reminder in part b) s/he has forgotten to convert the temperature into K and has then made a calculator error by multiplying by 95, instead of dividing by it. The candidate should realize that this is not a sensible figure for the answer and check his/her work.

Examiner's Answers

a) *pV = nRT* ✓ *where p = pressure in Pa , V = volume in m^3* ✓
n = number of moles of gas, R = universal gas constant,
T = temperature in K.
This is true only for gases which approximate to ideal gas behaviour.

b) *The temperatures must be in K because, for proportionality, the graph* ✓ *must be a straight line,* ***through the origin****. This happens for volume and pressure, only if the origin is shifted to where the volume or pressure line crosses the temperature axis i.e. –273 °C.* ✓

c) $\frac{pV}{T}$ = constant

so $\frac{101 \times 0.4}{288} = \frac{95 \times V}{298}$ ✓

$V = \frac{101 \times 0.4 \times 298}{(288 \times 95)}$ ✓ = 0.44l ✓

Typical Exam Question

a) State the conditions under which *p*, the pressure of a gas is proportional to *T*, the absolute temperature. **[2]**

b) A bottle of gas has a pressure of 200 kPa above atmospheric pressure at a temperature of 0° C.
The atmospheric pressure is 102kPa. Calculate the new pressure in the bottle when the temperature rises to 25° C. **[3]**

c) A mass of gas is at a pressure of 100 kPa and a temperature of 24° C. If it occupies a volume of 10litres, how many moles of gas are present? Take *R*= 8.31 J mol^{-1} K^{-1} **[3]**

a) *p is proportional to T for a fixed mass of gas* ✓ *, which approximates sufficiently closely to ideal gas behaviour. i.e. at low pressure* ✓

b) *p = 302kPa* ✓*, T = 273K*
New T = 298 K

$\frac{302}{273} = \frac{p}{298}$ ✓

$p = \frac{302 \times 298}{273}$ = 329.7kPa ✓

c) *pV = nRT*
$100kPa = 100 \times 10^3$ Pa and $10\ l = 10 \times 10^{-3} m^3$ ✓
$100 \times 10^3 \times 10 \times 10^{-3} = n \times 8.31 \times 297$ ✓

$n = \frac{1000}{8.31 \times 297}$ = 0.4 moles ✓

Questions

1 The table gives volumes of a fixed mass of gas at various pressures.

volume/litres	12	6	4	3	2.4	2
pressure/Pa	1	2	3	4	5	6

a) Plot a graph of pressure (*p*) against volume (*V*). Plot *p* on the x-axis.

b) This graph looks like a $y = \frac{1}{x}$ graph, suggesting that *V* is proportional to $\frac{1}{p}$, to confirm this plot a graph of *V* against $\frac{1}{p}$.

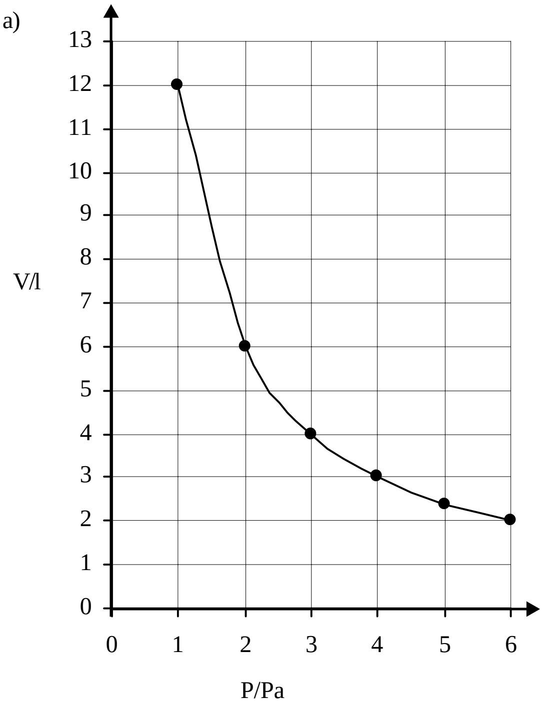
c) What feature of the graph confirms that *V* is proportional to $\frac{1}{p}$?

2. Two gas canisters contain different gases, but they are at the same temperature, pressure and volume. Use the ideal gas equation to show that each canister contains the same number of molecules.

3. A gas canister contains a mass of 64g of oxygen at a pressure of 105 kPa and a temperature of 22°C. What is its volume? [Take R = 8.3JK^{-1} mol^{-1}, relative molecular mass for oxygen is 32]

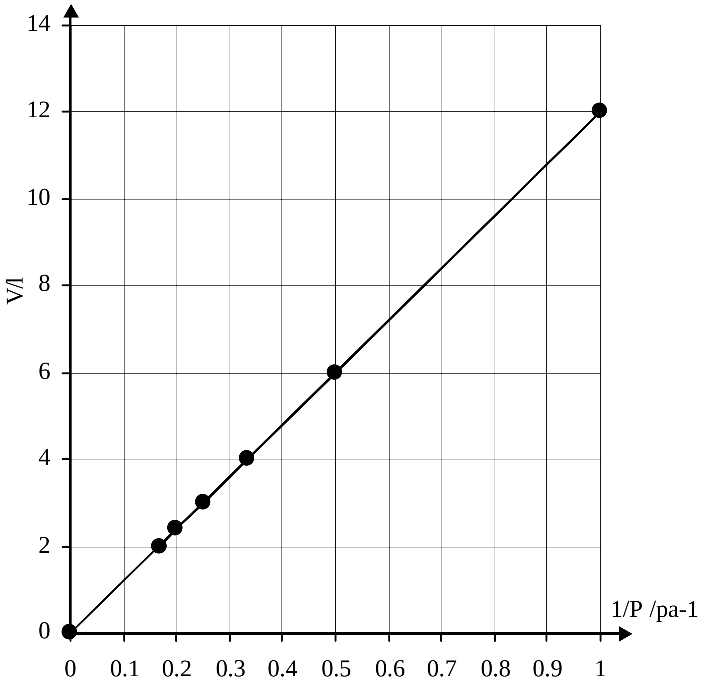
Answers

1. a)



b)

V/litres	12	6	4	3	2.4	2
$1/p$ /Pa^{-1}	1	0.5	0.33	0.25	0.2	0.17



c) The graph is a straight line through the origin, this confirms V is proportional to $1/p$.

2. The ideal gas equation is $pV = nRT$. If p,V and T are all equal for the two gases, then each must contain the same number of moles, since R is a universal constant. If each gas has the same number of moles, then each has the same number of molecules.

3. Since there are 64g of oxygen, the number of moles is 2.
$pV = nRT$
so $105 \times 10^3 \times V = 2 \times 8.3 \times 295$
$V = \dfrac{2 \times 8.3 \times 295}{105 \times 10^3} = 0.047 \text{ m}^3$

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Physics Factsheet

www.curriculum-press.co.uk Number 99

Gas Law Practicals and Absolute Zero

This factsheet will:
- show you practicals that allow you to arrive at the three Gas Laws
- show you how a straight line graph can verify a relationship
- explain how the concept of a minimum temperature (absolute zero) arises
- explain the Kelvin scale of temperature
- give you practice at exam-style questions

The bulk properties of gases are those that can be measured directly with no reference to the molecules of a gas. Three important bulk properties of gases are:
- pressure
- volume
- temperature

The pressure of a gas is the force per unit area it exerts on the walls. To understand the origin of this pressure we need to remember that the molecules of a gas are in constant motion. When they collide with the walls of the container, they rebound and suffer a change in ***momentum****. This change in momentum results in a* ***force*** *on the walls, and the force per unit area is the pressure.*

The **volume** of a gas affects its pressure. If we compress a gas (decrease its volume) at constant temperature, then:
- each molecule has less distance to travel on average before colliding with a wall of the container
- the number of collisions per second increases, even though the speed of the molecules is unchanged
- the momentum change per second (which is the same thing as force) increases
- the pressure increases

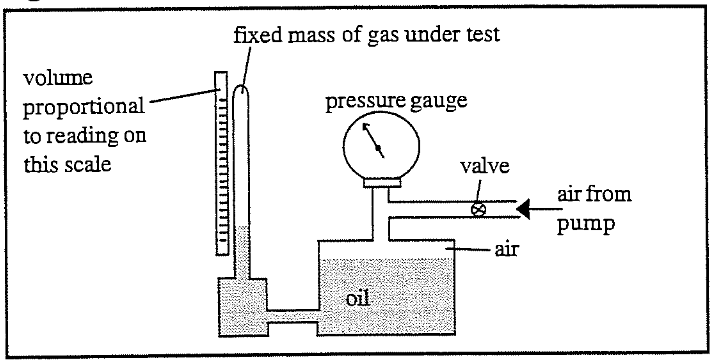
The **temperature** of a gas affects its pressure. If we heat a gas (increase its temperature) at constant volume, then:
- the average distance for each molecule between collisions is unchanged, but
- each molecule is travelling faster, so
- there are more collisions per second with the wall and each collisions transfers more momentum
- the force on the wall and hence the pressure increase.

These explanations are *qualitative*. To see *quantitatively* how the volume and temperature of a gas affect its pressure, we can perform two experiments which lead to results known as **Boyle's Law** and the **Pressure Law**.

Boyle's Law (relationship between pressure and volume at constant temperature)

The apparatus is shown in Fig 1. The gas being tested is compressed using the pump. Its pressure is read from the pressure gauge (because it is equal to the pressure of the air in the oil reservoir). The volume, V, is read from the scale.

Fig 1.



The pressure, p, is increased in stages, so that several pairs of values of p and V are taken.

A typical set of readings for the above experiment is reproduced below.

p/kPa	500	245	170	125	100
V/cm^3	1	2	3	4	5

Question 1: Plot a graph of p (y-axis) against V.

You will see that your graph curves. This shows that:
- the pressure of a gas *does* increase as its volume decreases, and
- that it does *not* decrease linearly.

However, it doesn't show how it *does* decrease, because there are many mathematical curves that look a bit like your graph (for example, at first glance you might think it was exponential).

Any textbook will tell you that the pressure and volume of a gas are related by:

$p \propto 1/V$, or pV = constant, provided the temperature is constant.

There are two ways you can test this relationship with your data.

Question 2
Firstly, add a row 'pV' to your table and calculate values. To what extent do your data support the theory that pV = constant?

Secondly, write $p \propto 1/V$ as p = constant × $1/V$, and compare it with the equation of a straight line $y = mx + c$

This shows that if we plot p (on the y-axis) against $1/V$, we should get a straight line through the origin (there is nothing corresponding to c, the y-intercept). Try it now!

Question 3
Add a row to the table of data and calculate values of 1/V. Now plot the graph. If you get a straight line through the origin, you have verified that p really is inversely proportional to 1/V (p ∝ 1/V, or pV = constant).

This neat trick of creating a straight-line graph crops up all over physics, and especially practical exams! What we have done so far only works, however, if you already suspect the relationship and want to test it.

However, we can do more! (Some exam boards at A2 expect you to). If you think that p and V may be related by a power law ($p \propto V$, $p \propto V^2$, $p \propto V^3$, $p \propto V^{-1}$, $p \propto V^{-2}$ etc) but you do not know which power, we can write the relationship as

> $p \propto V^n$, or $p = kV^n$
> (where k is the constant of proportionality)

Taking logs of both sides (base 10 or natural logs will both work) gives : $\log p = \log k + n \log V$

Comparing with $y = mx + c$ shows us that a graph of log p (y-axis) against log V should give a straight line with gradient 'n'. In other words, the gradient of the graph tells us the power!

> ***Question 4***
> ***Try this with the data for p and V given above. You should get a value of –1 for n, showing that $p \propto V^{-1}$, or $p \propto 1/V$***

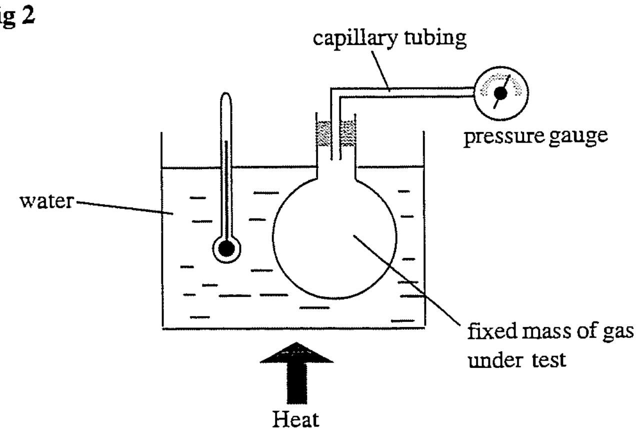
Boyle's Law is often written $p_1V_1 = p_2V_2$, where $_1$ and $_2$ denote initial and final values respectively.

> **Worked Example 1**
> ***A gas cylinder contains 0.040 m³ of nitrogen at a pressure of 2.0 MPa. When the gas is allowed to escape it expands until it reaches atmospheric pressure, 100 kPa.***
> ***(a) What volume does the gas occupy after expanding? [3 marks]***
> ***(b) What volume of nitrogen leaks from the cylinder? [1 mark]***
>
> **Answer**
> (a) $p_1V_1 = p_2V_2$ *[1]*
> $2.0 \times 10^6 \times 0.040 = 100 \times 10^3 \times V_2$ *[1]*
> $V_2 = 0.80\ m^3$ *[1]*
>
> (b) *0.040 m³ remains in the cylinder so 0.80 – 0.040 = **0.76 m³ escapes*** *[1]*

The Pressure Law (relationship between pressure and temperature at constant volume)

The apparatus is shown in Fig 2.

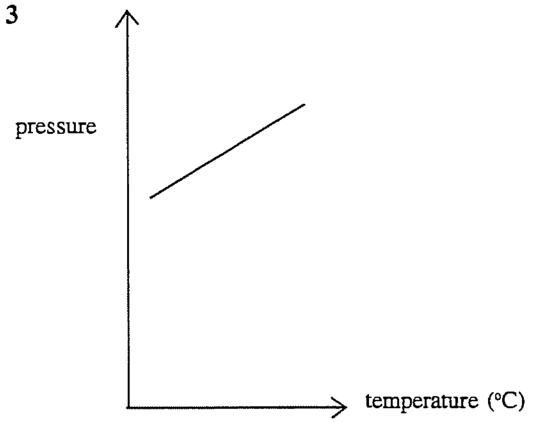
Fig 2



The water is heated, and pairs of values of pressure, p, and Celsius temperature, θ, are taken.

A graph of p against θ looks as shown in Fig 3:

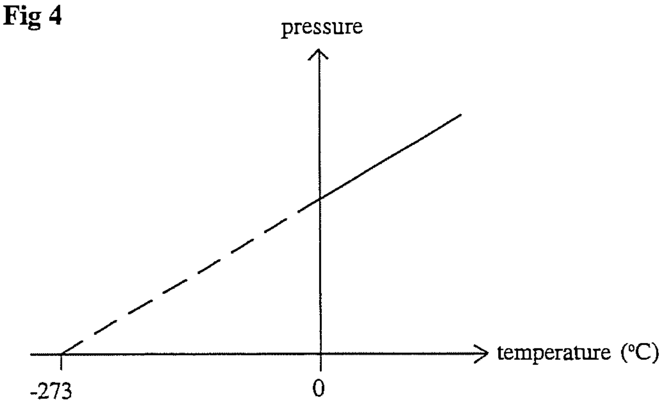
Fig 3



Absolute zero

If we assume that the gas from the Pressure Law experiment carries on behaving linearly at lower temperatures (this is a big assumption!), then we can extrapolate our graph, and we end up with Fig 4.

Fig 4



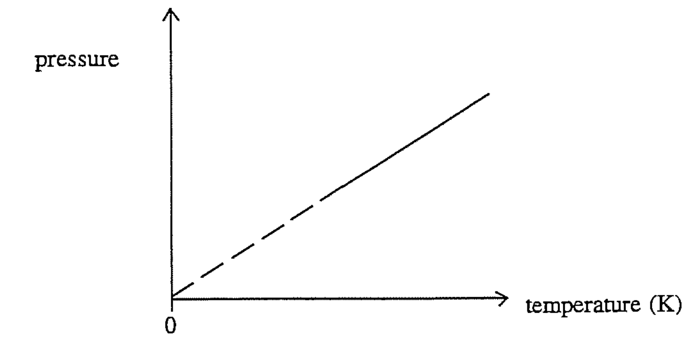
> - *The pressure of the gas becomes zero at a temperature of -273.15 °C.*
> - *Since the pressure is due to collisions from the moving molecules, a zero pressure suggests that the molecules are no longer moving.*
> - *The temperature of a gas is a measure of the kinetic energies of its molecules.*
> - *You can't have less kinetic energy than zero, which means that -273.15 °C must be the lowest temperature possible.*

It would make sense to have a scale of temperature in which the lowest possible temperature was at zero on the scale. It would also be nice if a *rise* of 1 degree on our new scale was equal to a rise of 1 °C. This is exactly what the **Kelvin Scale** of temperature does. We can change Celsius temperatures to Kelvin by adding 273.15:

$T / K = \theta / °C + 273.15$

The pressure-temperature graph then looks as follows in Fig 5.

Fig 5.



Notice that the pressure of a gas is directly proportional to temperature on the Kelvin scale.

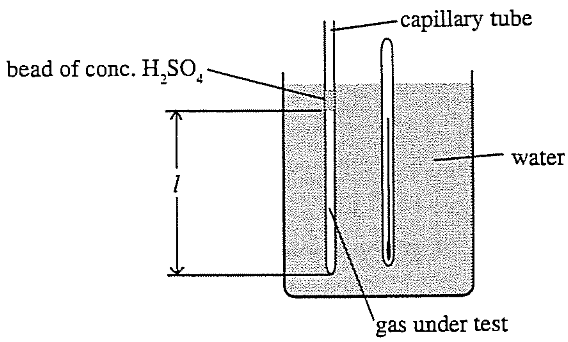
Thus, $p \propto T$ at constant V, provided we measure T in Kelvin. Another way of writing this is p/T = constant, or

$$\frac{p_1}{T_1} = \frac{p_2}{T_2}$$ where $_1$ and $_2$ denote initial and final values respectively.

Charles' Law (relationship between volume and temperature at constant pressure)

The apparatus is shown in Fig 6.

Fig 6.



The water is heated slowly and the length, l, of the air column (which is proportional to the volume) is measured at a number of Celsius temperature, θ. Concentrated sulphuric acid is used as it absorbs any moisture in the air. A graph of l against Celsius and Kelvin temperatures produces the graphs shown in Fig 7.

Thus, $V \propto T$ at constant p, provided we measure T in Kelvin. Another way of writing this is V/T = constant, or

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$ where $_1$ and $_2$ denote initial and final values respectively.

More about the gas laws

A gas which obeys all the gas laws perfectly is called an ideal gas. No ideal gas exists, but real gases behave like ideal gases quite well, especially at low pressure and high temperature.

Boyle's Law, Charles' Law and the pressure law can be combined into the following single relationship:

pV/T = constant, or

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$$ where $_1$ and $_2$ denote initial and final values respectively.

Pick one of the three variables. Keep it constant, and it will cancel from the equation, leaving you with one of the three Gas laws. But more than that, this relationship works when *none* of the three variables are constant.

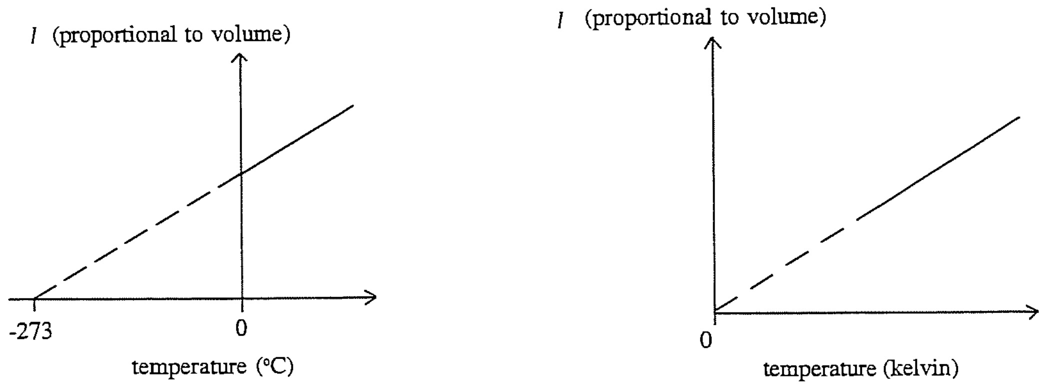
The ideal gas equation

Since pV/T is a constant, we can write pV/T = R. R has a value of 8.3 J mol^{-1} K^{-1} and is known as the molar gas constant. If we have n moles, then pV/T = nR, or pV = nRT, and this is known as the ideal gas equation.

'1 mole' is just a number we use to count atoms (just like 'a dozen' is a number we use to count eggs). 1 mole is the number of atoms in 12 g of carbon-12, and is approximately 6 x 10^{23}. The mass number of an isotope tells you roughly the mass of 1 mole. So if you want 1 mole of sodium-23, weigh out 23g of it.

***Exam Hint:** To use the ideal gas equation, we sometimes have to remember that number of moles = mass of gas / mass of one mole.*

Fig 7



Notice that the volume of a gas is proportional to its Kelvin temperature.

Practice Questions

1. (a) Write the following Celsius temperatures in Kelvin:
   (i) 100 °C
   (ii) 0 °C
   (iii) -196 °C

   (b) Write the following Kelvin temperatures in Celsius:
   (i) 300 K
   (ii) 600 K
   (iii) 3 K

2. A gas at STP (standard temperature and pressure = 0°C and 100 kPa) is enclosed in a rigid container. It is heated to 200°C. What is its new pressure? [3 marks]

3. A volume of gas is measured at 27.0 °C. What Celsius temperature will double the volume (with the pressure constant)? [4 marks]

4. A gas occupies 125 cm^3 at 120 kPa pressure and 40 °C. What will be its volume at 15 °C and 100 kPa pressure? [3 marks]

5. A container of volume 0.10 m^3 contains uranium hexafluoride (molar mass = 352 g) gas at a pressure of 1.0 MPa and a temperature of 27 °C. Assuming the gas is ideal, calculate the number of moles of gas present, and its mass. [5 marks]

Answers

1. (a) (i) 373.15 K
   (ii) 273.15 K
   (iii) 77.15 K

   (b) (i) 26.85 °C
   (ii) 326.85 °C
   (iii) –270.15 °C

2. $\frac{p_1}{T_1} = \frac{p_2}{T_2}$ [1]

   $\frac{(100 \times 10^3)}{273} = \frac{p_2}{473}$ [1]

   (don't forget to change the temperatures into Kelvin)
   $\mathbf{p_2}$ **= 173 kPa** [1]

3. $\frac{V_1}{T_1} = \frac{V_2}{T_2}$ [1]

   $\frac{V}{300} = \frac{2V}{T_2}$ [1]
   (remember to change the temperatures into Kelvin)
   $T_2 = 600\text{ K}$ [1]
   (make sure you can rearrange to get this – there is more than one way of doing it but you must be comfortable with one of them)
   = 327 °C [1]

4. $\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$ [1]

   $\frac{(120 \times 125)}{313} = \frac{(100 \times V_2)}{288}$ [1]

   Notice that with this equation we do not need to use SI units, so long as we use the same units on either side. However, we must use Kelvin and not Celsius temperatures.
   $V_2 = 138\text{ cm}^3$ [1]

5. $pV = nRT$ [1]
   $n = \frac{pV}{RT}$
   $= \frac{(10^6 \times 0.10)}{(8.3 \times 300)}$ [1]
   = **40 moles** [1]

   mass = mass of one mole × number of moles
   = 352 g mol^{-1} × 40 mol [1]
   = **1.4 × 10^4 g (14 kg)** [1]

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Physics Factsheet

January 2002 Number 25

Molecular Kinetic Theory

This Factsheet will explain:
- **the theory behind the kinetic model for gases;**
- **how the theory agrees with experimentally determined results;**
- **how to use the theory to do simple calculations.**

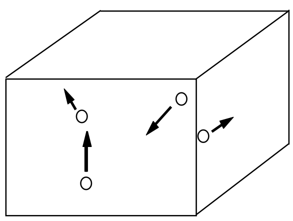
Before studying this Factsheet, you should be familiar with the simple qualitative ideas of how particles are arranged in solids, liquids, and gases.

You should know that:
- pressure = $\frac{\text{force}}{\text{area}}$
- momentum = mass × velocity
- kinetic energy = ½ mv^2.

The theory makes several assumptions to simplify the model:
- There are many molecules in the gas, so that average figures are representative of the whole.
- The molecules are moving rapidly and randomly.
- The behaviour of the molecules can be described by Newtonian mechanics.
- The molecules are far apart(compared until their diameters)and have no pull on each other.
- Any forces acting, do so only during collisions and then only for a very short time compared with the time between collisions.
- The collisions are elastic (kinetic energy is conserved).
- The total volume of the molecules is small compared to the volume of the gas as a whole.

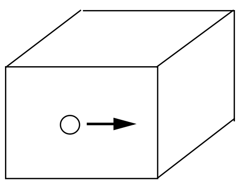
Exam Hint: *You will need to know these assumptions of the model.*

To understand the basis of the theory
Consider a rectangular box of gas:



The molecules are travelling in random directions with a range of speeds. The pressure of the gas on the walls of the box is caused by collisions of the molecules against the walls. A molecule's momentum changes when it hits the wall, (remember momentum is a **vector** quantity) and the force is given by the rate of change of momentum. The pressure is then the force per unit area.

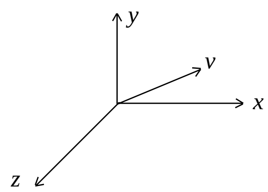
To derive an expression for the pressure



To simplify the ideas consider a molecule of mass m travelling parallel to the x axis with speed v_x in a cubic box of side l.
- The molecule's momentum is mv_x in the positive x direction.
- After an elastic collision with the wall, its momentum will be mv_x in the negative x direction, so the change of momentum is $2mv_x$.
- It will travel across the box and back, a distance of $2l$, at speed v_x, so it will take time $\frac{2l}{v_x}$ seconds.
- Thus the molecule will hit the face $\frac{v_x}{2l}$ times per second.

  Change of momentum per s $= 2mv_x \times \frac{v_x}{2l}$
  $= \frac{mv_x^2}{l}$

To generalise from this one molecule to average values for all the molecules, consider a molecule with speed v in a random direction.



It will have components v_x, v_y, v_z.
By Pythagoras: $v^2 = v_x^2 + v_y^2 + v_z^2$

Since the directions are random, there is no preference for any particular direction, so we can assume that on average for all the molecules:
$<v_x^2> = <v_y^2> = <v_z^2>$
so
$<v_x^2> = \frac{1}{3} <v^2>$

So the average change in momentum per s for n molecules on this one face, becomes: $\frac{1}{3} \frac{nm<v^2>}{l}$

pressure = $\frac{\text{force}}{\text{area}}$, so the pressure due to all the molecules

$$P = \frac{1}{3}\frac{nm<v^2>}{l^3}$$

$$P = \frac{1}{3}\frac{nm<v^2>}{V}$$ where V is the volume of the box.

RMS Speed.
The speed v_{rms} is the "root mean square" value of the speed. It is important to realise that the root mean square speed is **not** the same as the mean of the speeds. It is the square root of the mean of the squares of the speeds. Consider speeds of 1,2,3,4, and 5.

The mean speed $<v>$ is found by adding up 1 + 2 + 3 + 4 + 5 then dividing by 5 to give 3. The mean square speed $<v^2>$ is found by squaring the speeds = 1, 4, 9, 16, 25. Adding them up gives 55 and dividing by 5 gives 11. The r.m.s.speed is then the square root of 11 = 3.317 (3 D.P)

Exam hint: *A2 level questions often ask you to do this derivation in some form or other, often in words.*

This is an important result that we can consider in a number of ways:

1. $pV = \frac{1}{3} nm<v^2>$

The quantities on the right-hand side are all constant for a given gas at a given temperature, so this agrees with Boyle's Law that pV is a constant at a given temperature.

2. The density of the gas = $\frac{nm}{V}$ so we could write:

$p = \frac{1}{3}\rho <v^2>$ *where ρ is the density of the gas*

Worked example
The density of a gas at 20°C is 1.22kgm^{-3}. Assuming atmospheric pressure of 1.0 × $10^5 Nm^{-2}$, calculate v_{RMS} of the molecules.

$p = \frac{1}{3}\rho <v^2>$
$<v^2> = 3 \times 1 \times 10^5/1.22 = 2.46 \times 10^5$
$v_{RMS} = <v^2> = 496ms^{-1}$

3. The Universal Gas Law gives:

Universal Gas Law: pV = nRT, where n = the number of moles of gas, T = absolute temperature (°C + 273) Take care not to confuse this n for the number of moles, with the n used earlier for the number of molecules.

From our derivation:

$pV = \frac{1}{3} N_A m<v^2>$ [N_A is Avagadro's number, the number of molecules in 1 mole of gas.]

So

$\frac{1}{3} N_A m<v^2> = RT$

The kinetic energy of one mole of molecules is $\frac{1}{2} N_A m<v^2>$

so $E_K = \frac{3}{2} RT$ ($= \frac{1}{3}\times\frac{3}{2} N_A m <v^2>$)

i.e., the KE of the gas is directly related to the absolute temperature.

For the kinetic energy of 1molecule we must divide by N_A, which gives:

$E_K = \frac{3}{2}\frac{RT}{N_A}$

$= \frac{3}{2} kT$ where $k = \frac{R}{N_A}$,

k is called the Boltzmann constant.

K.E. of gases and molecules.
The K.E.of 1 mole of gas = $\frac{3}{2}$RT, where R is the molar gas constant.
The K.E. of 1 molecule = $\frac{3}{2}$kT where k is the Boltzmann constant (= $\frac{R}{N_A}$)

Worked example
A tank contains 2 moles of Neon (Relative Atomic Mass 20; i.e. the molar mass is 20g) at 20°*C*.
Calculate:
a) The total K.E. of the gas.
b) v_{RMS} of the molecules. *(take R = 8.3 $JK^{-1}mol^{-1}$)*

a) Total K.E $= \frac{3}{2} nRT$
[*Tip: Remember that T is the absolute temperature.*]
$= \frac{3}{2} \times 2 \times 8.3 \times 293$
$= 7.3 \times 10^3 J$

b) Total K.E. $= \frac{1}{2} nN_A m<v^2>$ *where $nN_A m$ is the number of moles × molar mass*

$<v^2> = 2 \times 7.3 \times 10^3/20 \times 10^{-3}$
$= 7.3 \times 10^5$
$v_{RMS} = 850\ ms^{-1}$ (2sf)

Typical Exam Question
a) State four assumptions of the Kinetic Theory of Gases. [4]

b) A gas molecule in a cubical box travels with speed v at right angles to one wall of the box. Show that the average pressure exerted on the wall is proportional to v^2. [4]

c) A container of volume 1.0 × $10^{-3} m^3$ has hydrogen at a pressure of 3 × $10^5 Nm^{-2}$ and temperature 20°C.
[Molar mass of hydrogen is 2g.
Take R = *8.3$JK^{-1}mol^{-1}$, N_A = 6.02 × 10^{23} mol^{-1}*]
Calculate i) the number of molecules in the container. [4]
ii) the v_{RMS} of the molecules. [3]

a) *Any four of:*
pressure = $\frac{force}{area}$
momentum = mass × velocity
kinetic energy = ½ mv^2
The collisions are elastic (kinetic energy is conserved).

b) *The molecules's momentum is mv, the average chanage in momentum when it hit a wall is 2mv.* ✓
It takes $\frac{2l}{V}$ to travel across the box and back, so it will hit a face $\frac{V}{2l}$ times per second. ✓
So change of momentum per second = $2mv \times \frac{V}{2l}$ ✓
$= \frac{mv^2}{l}$
pressure $= \frac{force}{area} = \frac{mv^2}{l^2} 2/l = \frac{mv^2}{l^3}$ ✓

c) i) $PV = nRT$
$n = 3 \times 10^5 \times 1 \times 10^{-3}/(8.3 \times 293)$ ✓
$= 0.12$ moles ✓
number of molecules
$= 0.12 \times 6.02 \times 10^{23}$ ✓
$= 7.43 \times 10^{22}$ ✓

ii) $pV = \frac{1}{2} nN_A m<v^2>$
$<v^2> = 2 \times 3 \times 10^5 \times 10^{-3} / (0.12 \times 2 \times 10^{-3})$ ✓
[$N_A m$ *is the molar mass in kg*]
$<v^2> = 6 \times 10^5 / 0.24 = 2.5 \times 10^6$ ✓
$v_{RMS} = 1900ms^{-1}$ ✓ (2SF)

Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's mark scheme is given below.

(a) How does Kinetic Theory explain the pressure exerted by a gas on the walls of its container? [3]

The molecules of the gas collide with the walls of the container. **1/3**

> The candidate should have realised that this is insufficient detail for 3 marks. The collisions cause a change in momentum of the molecules, and the force is the rate of change of momentum. Pressure is force per unit area.

(b) State and explain circumstances in which the Kinetic Theory expression for pressure in a gas $p = \frac{1}{3}\rho\langle v^2\rangle$ does not hold. [2]

At low temperature and pressure. **1/2**

> Insufficient detail. It will not hold if the conditions mean that any of the assumptions of the theory are invalid. e.g. for small quantities of gas there may not be enough molecules to justify statistical averages. At very low volume and high pressure the assumption that the volume of the molecules themselves is much less than the volume of the container may not be valid.

(c) A pressurised tank at 25°C contains 2 moles of helium gas (Molar mass of helium = 4g). Take R = 8.3$JK^{-1}mol^{-1}$
Calculate i) the total K.E. of the gas. [2]

i) $K.E. = \frac{3}{2}RT = \frac{3}{2} \times 8.3 \times 25 = 311J$ **1/2**

> Two common mistakes – the candidate has forgotten that there are 2 moles of gas and has failed to convert the temperature to absolute temperature.

ii) V_{RMS} for the molecules. [3]

ii) $K.E. = ½ \times nN_A m\langle v^2\rangle$
$311 = ½ \times 4 \langle v^2\rangle$
$\langle v^2\rangle = 311 \times 2/4$
$v_{RMS} = 12.5\ ms^{-1}$ **2/3**

> The candidate has already been penalised for the incorrect calculation for the K.E. and for omitting the number of moles, so that would be "error carried forward" (e.c.f), but s/he has now forgotten to express the molar mass in kg. S/he should have realised that this value is far too low for a v_{RMS}.

Examiner's answers

a) *Pressure is force per unit area.✓ Force is rate of change of momentum.✓ The force of the molecules against the walls arises from the change in momentum which occurs when molecules collide elastically with the walls. ✓*

b) *The expression will not hold if any of the assumptions of the theory is invalid.✓ e.g. if the volume of the container is not large compared with the volume of the molecules themselves.✓*

c) i) $KE = \frac{3}{2}nRT$ ✓
$= \frac{3}{2} \times 2 \times 8.3 \times 298$ ✓
$= 7420J$ ✓

ii) $KE = ½ nN_A m\langle v^2\rangle$
$= ½ \times 2 \times 4 \times 10^{-3}\langle v^2\rangle$ ✓
$\langle v^2\rangle = 7420/4 \times 10^{-3}$ ✓
$= 1.86 \times 10^6$ ✓
$v_{RMS} = 1\text{-}36 \times 10^3 ms^{-1}$ ✓
$v_{RMS} = 854\ ms^{-1}$

Questions

1. Use the ideas of Kinetic Theory to show that $\langle v^2\rangle$ is proportional to the absolute temperature.

2. Explain what is meant by v_{RMS} for the molecules of a gas.

3. A gas exerts a pressure of $2 \times 10^5 Pa$. The v_{RMS} is found to be $1100ms^{-1}$. What is the density of the gas?

4. i) Calculate the temperature at which hydrogen molecules have an RMS speed of $1200ms^{-1}$. $R = 8.3JK^{-1}.mol^{-1}$. (Molar mass of hydrogen = 2g.)

   ii) Calculate the KE of 1 mole of hydrogen gas at this temperature.

   iii) Calculate the KE of 1 molecule of hydrogen at this temperature. The Boltzmann constant $k = 1.4 \times 10^{-23}$.

Answers

1. See text.

2. v_{RMS} is the square root of the mean value of the speeds of the molecules.

3. $P = \frac{1}{3} p\langle v^2\rangle$
$p = 3 \times 2 \times \frac{10^5}{1100^2}$
$= 0.50kgm^{-3}$

4.i) $RT = \frac{1}{3} N_A m\langle v^2\rangle$
$T = \frac{1}{3} \times 2 \times 10^{-3} \times \frac{1200^2}{8.3}$
$= 116K$

ii) Energy of 1mole $= \frac{3}{2}RT$
$= \frac{3}{2} \times 8.3 \times 116$
$= 1.4 \times 10^2 J$

iii) Energy of 1 molecule $= \frac{\text{energy of 1 unit}}{\text{number of molecule per mole}}$
$= \times \frac{1.4 \times 10^{-2}J}{N_A}$
$= \frac{1.4 \times 10^{-2}J}{6.02 \times 10^{23}}$
$= 2.3 \times 10^{-23}J$

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First Law of Thermodynamics

This Factsheet is about the first law of thermodynamics, which is concerned with how energy is transferred from and to any material. In order to understand the first law of thermodynamics we must first consider the quantities used in expressing it.

A Note of Caution
The sign convention for the terms in the first law of thermodynamics that have been used in this factsheet are consistent with AQA specification B and the Edexcel specifications. The AQA specification A unit 5 option of applied physics defines work done on a gas as negative and the work done by the gas as positive. This means that the format of the first law of thermodynamics takes the form: $\Delta U = Q - W$. Please check that you are using the correct quantitative form of the first law of thermodynamics with your specification.

Heat
When two objects, at different temperatures are placed next to each other there will be a transfer of heat.

The symbol used for heat is Q.
*Q is a **positive** number when heat is **supplied** to an object.*
*When an object **loses** heat then Q is **negative**.*

***Exam Hint :** When asked to define heat or state what you understand by heat , it should always be mentioned that heat supplied to the system are considered to be positive values.*

Heat will be transferred from the hotter object to the cooler object. If we concentrate on just one of those objects, energy supplied to it by heating has two possible consequences: - a change in its temperature and/or a change of state, from liquid to gas for example.

Heat supplied to an object or removed from it can lead to a change in temperature of the object or a change of state of the object, or even a combination of both of these things.

Specific Heat Capacity
If an object has heat supplied or removed such that it results in a change in temperature then the size of the change in temperature will be governed by the **specific heat capacity** of the object. This is the amount of heat required to increase the temperature of 1 kg of the material by 1 °C.

Specific heat capacity is the amount of heat energy needed to increase the temperature of 1 kg of the material by 1 °C.

$$c = \frac{Q}{m\Delta\theta}$$

Q = heat energy supplied (J)
c = specific heat capacity ($Jkg^{-1}\,{}^oC^{-1}$ or $Jkg^{-1}K^{-1}$)
m = mass of material (kg)
$\Delta\theta$ = change in temperature (°C)
= final temperature - inital temperature

If the temperature of the substance is increasing then heat is being supplied to the substance and the change in heat , Q, is positive. If the substance is cooling down then it is losing energy and the change in heat energy, Q, is negative.

Specific latent heat
If an object has a change in energy that results in a change of state, from solid to liquid, liquid to gas or vice versa, then the amount of mass changing from one state to another is governed by **specific latent heat**. Every material has two values of specific latent heat, one value for a change of state from a solid to a liquid, called the specific latent heat of **fusion**, and one value for a change of state from a liquid into a gas, called the specific latent heat of **vaporisation**.

If a substance is changing from a solid to a liquid or from a liquid to a gas then energy is being supplied to a substance and Q, is **positive**.

If a substance is changing from a gas to a liquid or a liquid to a solid then it is losing energy and the change in heat energy, Q, is **negative**.

Specific latent heat of fusion is the amount of energy required to turn 1 kg of a solid into a liquid.

Specific latent heat of vaporisation is the amount of energy required to turn 1 kg of a liquid into a gas.

$$l = \frac{Q}{m}$$

l = specific latent heat are (Jkg^{-1})
Q = heat energy supplied (J)
m = mass of material changing state (kg)

Typical Exam Question
A kettle is filled with 2.0kg of water.
(a) Calculate how much energy is required to increase the temperature of the water from 20 °C to boiling point, 100 °C. [3]
(b) If the kettle is left running so that it supplies another 12000J of energy, how much of the water will have evaporated and turned from a liquid into a vapour. [2]
Specific heat capacity of water = 4200 $Jkg^{-1}\,{}^0C^{-1}$
Specific latent heat of vaporisation of water = 2.3 x 10^6 Jkg^{-1}

(a) A straight substitution of figures into our equation for specific heat capacity is all that is required. The equation has been rearranged to make the change in heat energy, Q, the subject of the equation. Care must be taken to include the change in temperature of the water. Change in temperature:
$\Delta\theta$ = final temperature – initial temperature = 100 – 20 = 80 °C ✓
Heat energy supplied, $Q = mc\Delta\theta = 2 \times 4200 \times 80$ = 672,000 J ✓ ✓

(b) Again, to answer the question, all that is required is the substitution of values into the equation for specific latent heat. The equation has been rearranged to give mass as the subject of the equation.
Mass evaporated:
$$m = \frac{Q}{l} = \frac{(12000)}{(2.3 \times 10^6)} = 0.0052kg = 5.2g$$ ✓ ✓
As would be expected, this is not a large mass of water
Note how, in each of these equations, the values for energy are positive as energy has been supplied to the water.

Work Done

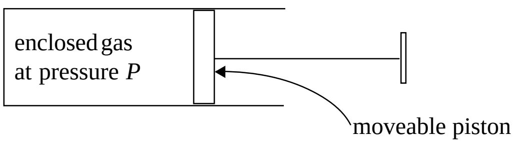
Work is done on a material whenever a force that is being applied to the material moves through a distance. An example of work being done on a solid is when a force is being used to move one object across another object against friction. In this instance, the work done on the solid would cause the two solid objects to increase in temperature. Work can also be done on fluids such as a gas being compressed. When the piston of a sealed bike pump is pushed in, as the force moves through a distance it will increase the temperature of the gas as well as store some energy in the gas. This stored energy could be released by letting go of the piston. The piston would spring back out as the gas expands.

> *Work is done when a force moves through a distance in the direction of the force.*
>
> $W = Fx$ 　 *W = work done (J)*
> *F = force applied (N)*
> *x = distance moved (m)*

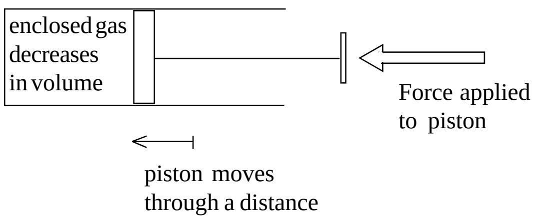
When work is done, energy is transformed from one form to another. Work done has the same units of energy, joules (J).

Work done and gases

The work done on a gas is also related to the pressure and volume of a gas. Consider a gas that is enclosed in a cylinder. The pressure of the gas inside the cylinder is *P*. One wall of the cylinder is moveable like a piston. A side view of this situation is shown below.



Now, imagine a force is exerted on the moveable wall that squashes the gas a little, moving the piston a small distance. The force has moved through a distance, doing work on the gas.



The amount of work done on the gas is given by its pressure multiplied by its change in volume.

> *The work done on a gas is given by the following equation:*
>
> $W = p\Delta V$ 　 *W = work done (J)*
> *P = pressure (pa)*
> *ΔV = distance moved (m)*
>
> *The pressure of the gas is assumed to* ***stay the same*** *when using this equation.*

- When work is being done **on** a gas (in other words it is being **compressed**), work done is given a **positive** sign.
- When work is being done by a gas, (in other words the gas is **expanding** and pushing back its boundaries), it is given a **negative** sign.

Note how the symbol, Δ, which is the greek letter delta, is used to represent a change in volume. This is in the same way as Δθ represents a change in temperature.

Typical Exam Question

Air is enclosed in a container that has a moveable piston. The container has an initial volume of 7.50 x 10^{-4} m^3. A constant force of 125N is applied to the piston so that it moves a distance of 4.50cm, compressing the gas. The pressure of the gas remains at 100,000Pa throughout the compression.

(a) Calculate the work done in moving the piston. [2]

(b) Calculate the final volume of the gas, stating any assumption that you make. [4]

(a) A force and distance have been given in the question so that these numbers can be substituted into our general equation for work done. Care must be taken to change the units of distance into metres before substitution into the equation. The final answer has also been quoted to the same number of significant figures as given in the question.

$W = Fx = 125 \times 0.045$ ✓

$= 5.625 = 5.63J$ ✓

(b) The question provides a pressure for the compression, and a value for work done has just been calculated. We can use these values to calculate the change in volume of the gas. Note how the value for work done calculated in the first part of the question has a positive value as it represents work being done on the gas. As the question suggests, an assumption has been made that all the work done in pushing the piston has been transferred to the gas. Note how all of the significant figures from the last calculation have been used when substituting into this equation.

$W = p\Delta V$

$\Delta V = \frac{W}{p} = \frac{5.625}{100{,}000}$ ✓ $= 5.625 \times 10^{-5} m^3$ ✓

Knowing the change in volume and the initial volume, the final volume can now be calculated.

Final volume = initial volume - change in volume

$= 7.5 \times 10^{-4} - 5.625 \times 10^{-5} = 6.9375 \times 10^{-4} = 6.94 \times 10^{-4} m^3$ ✓

Assumption: All of the work done in pushing the piston is transferred to work done on the gas. ✓

Internal Energy

Internal Energy is the term used to describe all of the energy contained in a material and it is given the symbol, U. Any change in the internal energy of a material is given the symbol, ΔU. There are two forms of energy that can make up the internal energy of a material:

(i) **Potential energy**. This is the energy stored in the stretched or compressed bonds between the molecules of a material.

(ii) **Kinetic energy**. This is the energy that the molecules in a material have because they are moving. In a solid, this movement is vibrational movement centred around a fixed position. In a liquid or a gas this movement is linear as the molecules travel in straight lines in between collisions with other molecules or the sides of the container.

- In solids there is a roughly equal split between the potential and kinetic energy.
- In a liquid, the majority of the internal energy is made up of the kinetic energy of the molecules although there is still some potential energy.
- In an ideal gas, there are no forces of attraction between the molecules, so there is no potential energy and all of the internal energy is made up of the kinetic energy of the fast-moving gas molecules.

> *The internal energy of a material is the total sum of all of the energy contained in the material. For solids and liquids this will be the sum of the potential energy and kinetic energy of the molecules. For a gas, there is no potential energy and the internal energy is the sum of the kinetic energy of the molecules.*

***Exam Hint**: To avoid making errors with units of values used in calculations it is best to change all units into standard units before starting any calculation. Some common standard units used in this topic are: Energy – Joules (J), time – s, volume – m^3, pressure – pascals (pa)*

The First Law of Thermodynamics

The three concepts that have so far been covered by this Factsheet - energy transferred by heating, work done and internal energy - are linked by the first law of thermodynamics.

A change in the internal energy of a substance will occur if it gains or loses heat energy and also if the substance receives or does work on its surroundings or has work done on it.

$\Delta U = Q + W$ *ΔU = **increase** in internal energy of the substance (J)*
*Q = heat energy **supplied** to the substance (J)*
*W = work done **on** the substance (J)*

Hence, supplying the substance with heat energy, making Q positive will increase the internal energy of the substance and ΔU will be positive. Also, doing work on the substance, making W positive, will also increase the internal energy of the object, again making ΔU positive.

Similarly, if energy is removed from a substance and it is allowed to cool, making Q negative, it will experience a decrease in internal energy and ΔU will be negative. Also if the substance does work on the surrounding, making W negative, it will experience a decrease in internal energy, and ΔU will be negative.

A simple example in the use of this equation would be to consider a cleaner scrubbing the inside of a fridge. As he scrubs, he is applying a force through a distance and performing 50J of work on the interior of the fridge. At the same time the interior of the fridge is being cooled and has had 30J of heat energy removed. If we consider the change in internal energy of the interior of the fridge by using the first law of thermodynamics we must first decide the sign of the change in heat energy and work done.

The heat energy supplied to the fridge interior, Q = -30J

The work being done by the fridge interior, W =+50J

Increase in internal energy of the fridge interior;
$\Delta U = Q + W = -30 + 50 = 20J$

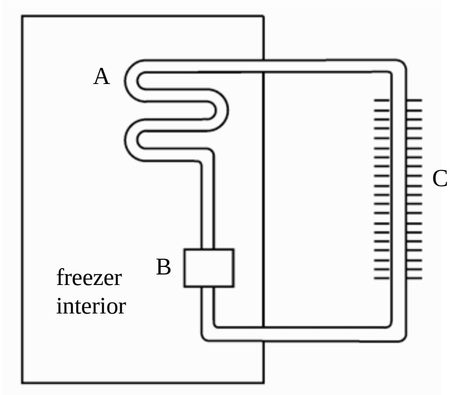
This increase in internal energy would be seen as an increase in the temperature of the fridge interior.

***Exam Hint**: Decide and write down the signs of work done and energy before using the values in the first law of thermodynamics.*

Typical Exam Question

(a) The First Law of thermodynamics can be stated in the form: ΔU = Q+ W . State what you understand by the term ΔU in this law. [1]

(b) The diagram below shows a freezer. The fluid inside the pipes evaporates at A, is compressed by the pump at B and cools as it passes through the pipes at C.



(i) Explain where the fluid is gaining or losing heat energy in the diagram. [2]
(ii) Explain where the fluid is doing work or is having work done on it in the diagram. [2]

(c) 3.5 kg of water with a temperature of 20°C is placed in the freezer. It takes 2.0 hours to freeze the water without cooling it below 0°C.
(i) How much energy is removed from the water? [5]
(ii) What is the power of the freezer. [2]
Specific latent heat of fusion for water = 3.4 x 10^5Jkg^{-1}

(a) ΔU is the change in internal energy of the substance. A positive value for ΔU represents an increase in internal energy. ✓

(b) (i) The fluid is gaining latent heat energy inside the freezer, at A, ✓ *as it changes from a liquid to a gas. The fluid loses heat energy at C where it cools down.* ✓

(ii) The fluid is having work done on it where it is being compressed in the pump at B. ✓ *The fluid is doing work at A where it is expanding.* ✓

(c) (i) This part of the question requires two initial calculations.

One to calculate the energy removed because the water is cooling down and one to calculate the energy removed because the water is changing into a solid.

Considering the cooling process first we can use our equation for specific heat capacity, rearranged to make energy the subject of the equation. The change in temperature is from 20 °C to 0 °C.
$Q = mc\Delta\theta = 3.5 \times 4200 \times 20$ ✓ $= 294{,}000$ J ✓
Now we can calculate the energy removed from the water to change it into a solid using our equation for latent heat.
$Q = ml = 3.5 \times (3.4 \times 10^5)$ ✓ $= 1190000$ J ✓
The total energy removed from the water is the sum of these.
Total heat energy removed
$= 294{,}000 + 1190000$ ✓
$= 1484000$
$= 1.5 \times 10^6$ J ✓ *(to 2 significant figures as given in the question).*

(ii) To calculate the power of the freezer, we need to use the equation
$Power = \frac{Energy}{Time}$
remembering to change the time into seconds.
$P = \frac{E}{t} = \frac{1.5 \times 10^6}{2083}$ ✓ $= 2.1$ kW ✓

Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answer and how they can be improved. The examiner's answer is given below.

(a) The first law of thermodynamics may be written as: $\Delta U = Q + W$. State what you understand by each of the terms in the equation. [3]

ΔU is the change in internal energy
Q is heat energy
W is work done ✓ ✗✗ 1/3

> No direction has been stated. Each term should have stated which direction of energy transfer is considered positive

(b) 10 g of water occupying a volume of 10 cm³ is heated in a kettle from 20 °C to 100 °C and then evaporated into steam, which occupies a volume of 0.0016 m³.

(i) Calculate the heat energy required to increase the temperature of the water to 100°C.
Specific heat capacity of water = 4,200 Jkg^{-1o}C^{-1} [2]

$Q = mc\Delta\theta$ ✓ $= 10 \times 4200 \times 100 = 4{,}200{,}000J$ ✗ 1/2

> The student has not changed the mass of the water into kg and the final temperature has been used the calculation, not the change in temperature – this scores 1/2 for the correct equation being used.

(ii) Calculate the heat energy required to turn the water at 100°C into steam. Latent heat of vaporisation for water = 2.3×10^6Jkg^{-1} [2]

$Q = ml$ ✓ $= 10 \times 2.3 \times 10^6 = 2.3 \times 10^7 J$ ✗ 1/2

> Once again, the incorrect mass has been used and again scores 1/2 for the correct equation being used

(iii) What is the total heat energy supplied to the water? [1]

Total energy $= 4{,}200{,}000 + 2.3 \times 10^7 = 2.72 \times 10^7 J$ ✓ 1/1

> Although incorrect values for these energies have been used, the student would receive this mark as his calculation is correct, using errors that have been carried forward.

(c) Calculate the work is done by the steam as it expands. [2]

$W = p\Delta V$ ✓ $= 101{,}000 \times 0.0016 = 161.6J$ ✓ 2/2

> A correct answer scoring full marks

(d) Calculate the increase in internal energy of the water for the entire process. [3]

$\Delta U = Q$ ✓ $+ W = 2.72 \times 10^7 + 161.6 = 27{,}200{,}161.6 J$ ✗✗ 1/3

> The sign of the work done is incorrect as the gas is doing work. The answer has also been quoted to an inappropriate number of significant figures again this scores a single mark for the correct equation being used.

Examiner's Answers

(a) *ΔU is the increase in internal energy a substance.*
Q is energy supplied to the substance by heating.
W is work done on the substance.

(b) (i) $Q = mc\Delta\theta = 0.01 \times 4200 \times (100-20) = 3360J$ (quote 3.4×10^3 J)
(ii) $Q = ml = 0.01 \times (2.3 \times 10^6) = 23{,}000J$ (quote 2.3×10^4 J)
(iii) Total heat energy supplied = 3360 + 23,000 = 26,360 = 26,400J (quote 2.64×10^4 J)

(c) $W = p\Delta V = 101{,}000 \times 0.0016 = 161.6J$ (quote 1.6×102 J)

(d) $\Delta U = Q + W = 26360 - 161.6 = 26{,}198.4 = 26{,}200J$

Qualitative (Concept Test)

1. What is specific heat capacity?
2. What is the difference between latent heat of vaporisation and latent heat of fusion.
3. When a substance changes from a liquid to a solid, does it receive heat energy or give out heat energy?
4. What sign, positive or negative, is given to the value for the work done by a gas that expands?
5. What two types of energy generally make up the internal energy of a substance? What type of substance only contains one of these types of energy and which is it?
6. What do the three terms represent in the first law of thermodynamics?

Quantitative (Calculation Test)

1. How much heat energy will be supplied to 1.5kg of water when it is heated from 20°C to 80°C? Specific heat capacity of water = 4,200Jkg^{-1o}C^{-1}.
2. How much heat energy will have to be supplied melt to 0.5kg of ice? Specific latent heat of fusion of ice = 3.4×10^5 J kg^{-1}
3. (a) A gas is allowed to expand at a constant pressure of 150,000Pa from 5m³ to 15m³. What work has been done?
   (b) If the gas also receives 3×10^6 J of energy by heating what is the increase in internal energy of the gas?
4. A man drags a crate across the floor of his refrigerated lorry. The crate loses 150J of internal energy and has 350J of energy removed from it by the cold surroundings.
   (a) Calculate the work done on the crate.
   (b) If the force exerted on the crate is 50N, calculate the distance moved by the crate.

Qualitative Test Answers

The answers can be found in the text

Quantitative Test Answers

1. $Q = mc\Delta\theta = (1.5)(4200)(80\text{-}20) = 378{,}000J$
2. $Q = ml = (0.5)(-3.4 \times 10^5) = -1.7 \times 10^5J$. Note how the answer is negative as the ice is giving out energy and not supplying it.
3. (a) $W = p\Delta V = (150000)(5\text{-}15) = -1{,}500{,}000J$. Again, note the minus sign showing that the gas has been doing work.
   (b) $\Delta U = Q + W = 3 \times 10^6 – 1{,}500{,}000 = 1{,}500{,}000J$
4. (a) $W = \Delta U – Q = -150 –(-350) = 200J$
   (b) $x = W/F = (200)/(50) = 4m$

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Physics Factsheet

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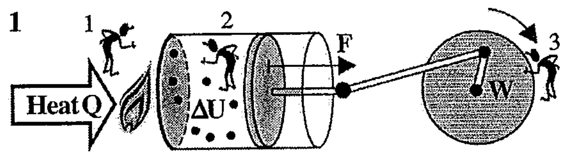
Calculations: First Law of Thermodynamics

<u>The first law of thermodynamics</u> can be explained by Fig 1 which shows a frictionless cylinder and piston containing a quantity of gas.

Fig 1



$Q = \Delta U + W$

Three imps are watching what is happening. The first is aware of a quantity of heat (Q) entering the cylinder, the second imp doesn't understand 'heat' but sees the gas molecules speed up while the third sees work output (W) at the turning wheel. All three imps understand the term *energy.* They decide that the heat *energy* input must equal the *energy gained* by the gas molecules plus the *energy* used when the work is done.

So, if 100J of heat enter the gas and 70J of energy are given to the gas molecules then the remaining 30J must appear as the work done.

Molecular kinetic energy and Temperature

The second imp is keeping an eye on the gas molecules. He sees the molecules speed up and *gain kinetic energy.* When you or I look at the gas we see that there is a *gain in temperature.*

> *When the temperature increases we deduce that the molecular kinetic energy has also increased.*
>
> *We use U to represent the kinetic energy of all the molecules and T for temperature and write $U \propto T$*
> *Remember that the temperature must be in kelvin K.*

> ***Example 1***
> ***A gas has a temperature of 27°C. Find the temperature when the kinetic energy of the gas is doubled.***
> *Answer. The temperature $T = 27 + 273 = 300K$. To double the kinetic energy we must double the kelvin temperature. So the required temperature is 600K or $(600 - 273) = 327^oC$.*

Another name for the kinetic energy of the gas molecules is the *internal energy....* this term is used in thermodynamics. In the example above, the internal energy is doubled and in question 1 (at the end), the gas loses 10% of its internal energy.

Remember that the second imp 'sees' a change in internal energy while you and I 'see' a change in temperature.

Now look again at Fig 1.
The heat that flows into the gas has two jobs to do.
1) it must warm the gas ie increase the internal energy and
2) it must provide for work to be done

> *We write $Q = \Delta U + W$. This is the first law of thermodynamics*
> *Q is the heat supplied **to** the system, ΔU is the **rise** in internal energy of the system, and W is the work done **by** the system.*

In our first example, the heat Q = 100J, the increase in internal energy ΔU = 70J and the work done W = 30 J

This illustrates an important point about heat and work. If we rub our hands together, work is done and heat produced. In this example, *all* of the work done against friction is converted into heat.

Work → Heat (100% conversion)

If we try the *reverse* process, Heat → Work, we find that 100% conversion is impossible. This is not due to quality of workmanship in making the cylinder and piston but is down to the laws of physics! Some of the heat supplied is used to warm the gas (ΔU). Only that left over is available for work to be done.

> *Example 2*
> ***For the example in fig 1 (Q = 100J, ΔU = 70 J and W = 30J), calculate the efficiency of the cylinder and wheel arrangement.***
> *In this example we are trying to convert the input (100J of heat) into the output (30J of work).*
>
> $$efficiency = \frac{output}{input} = \frac{30J}{100J} = 30\%$$

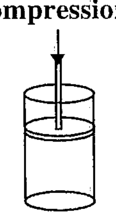
Adiabatic Changes

You may have noticed when pumping up a tyre, that the pump gets quite warm. This is not just due to friction but because you are *compressing* the gas. You are doing work on the gas.

Imagine that the cylinder in Fig 2 is perfectly insulated so that heat cannot enter nor leave the gas.

When the piston is rapidly pushed inwards, there is a temperature rise in the gas. How can this be? The temperature increases but heat is <u>not</u> supplied.

Fig 2. rapid compression



The answer is found from the first law, Q = ΔU + W.
Suppose that 200J of work are done in *compressing* the gas.

> *Because the gas is **compressed** we must take the work done to be **negative.** Heat doesn't enter or leave the gas so Q = 0. Substitute in the equation:-* $Q = \Delta U + W$
> $0 = \Delta U + (-200J)$
> $200J = \Delta U$
> *The internal energy has increased by 200J*

Remembering that internal energy depends on temperature, we see that the temperature must also increase.

This is an example of an **ADIABATIC CHANGE**, ie, where **heat is *not* allowed into or out of the gas.**

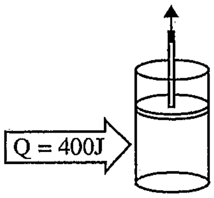
For a compression, there is **no** heat exchange but there **is** a temperature rise! Put another way, all the work done goes to increase the internal energy (and the temperature) of the gas.

Isothermal Changes

Look at Fig 3 showing a gas expanding at a ***constant temperature.*** This is an **Isothermal Change.**

Because the temperature doesn't change, the internal energy doesn't change. ie $\Delta U = 0$

Fig 3. slow isothermal expansion



This time 400J of work are done.
Because the gas is ***expanded*** we must take the work done to be ***positive.***

Substitute in the equation

$Q = DU + W$
$Q = 0 + 400J$
$Q = 400J$

For the change to be isothermal, it must take place slowly to allow heat into or out of the gas. We have just shown that because 400J of work are done isothermally, the heat supplied must *also* be 400J.

Summary

The First Law of Thermodynamics says $Q = \Delta U + W$ where Q is the heat absorbed, ΔU is the increase in internal energy and W is the work done by the system.

When the gas expands W is positive; for a compression W is negative. If the heat enters the gas, Q is positive. If the heat leaves the gas, Q is negative. $U \propto T$ (kelvin temperature). If U increases, then the temperature increases (and vice versa).

Work done by a gas

Fig 4a shows some gas trapped in a cylinder by a piston.

Fig 4a

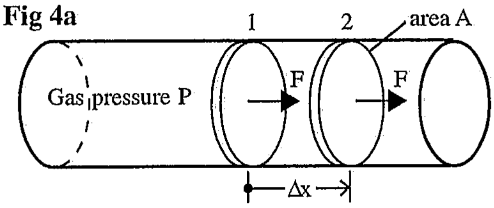
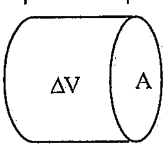


Fig 4b



$\Delta V = A \times \Delta X$

As the gas expands, the force F pushes the piston through a small distance, so we conclude that **work is done!** We have to find the work done by the gas as the piston moves a small distance Δx from position 1 to position 2.

The gas is at a pressure P and exerts a force F on the piston with area A.

$$\text{Pressure} = \frac{\text{force}}{\text{area}} \quad P = \frac{F}{A} \Rightarrow F = P \times A$$

The distance Δx is small so the pressure remains constant. The work done by the force F is also small and is ΔW.

Work done = Force × Distance, so
$\Delta W = F \times \Delta X$
$\Delta W = (P \times A) \times \Delta X = P \times (A \times \Delta X)$

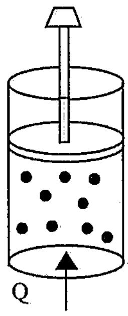
From Fig 4b we can see that the *increase* in volume $\Delta V = A \times \Delta X$
Hence we have the important result

$$\Delta W = P \times \Delta V$$

You should check the units of this equation. The units of P are Nm^{-2} ; those for ΔV are m^3 .
Using the equation above you should show that the unit for ΔW is the Joule.

Example 3
Fig 5 shows a gas being heated at a constant pressure of 10^5 Pa (atmospheric). The volume of the gas increases by 0.5ℓ. Find the work done.

Fig 5



When we use $\Delta W = P \times \Delta V$, we want W in JOULES.
*This means that P **must** be in Pa (Nm^{-2})*
*and **V must** be in m^3.*
Because 10^3 litres = $1m^3$, $0.5\ l = 0.5 \times 10^{-3}\ m^3$
so we write $\Delta V = 0.5 \times 10^{-3}\ m^3$.

$\Delta W = P \times \Delta V$
$\Delta W = 10^5 \times (0.5 \times 10^{-3})$
$\Delta W = 50\ J$

The work done by the expanding gas is 50 J.

Remember that pressure is sometimes given in kPa or MPa, so you must then introduce 10^3 or 10^6 respectively in your calculation.

Exam Hint: *Remember to convert the volume into m^3 and the pressure into Pa (Nm^{-2}) before performing the calculation.*

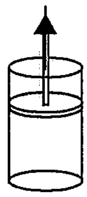
If in this case, (Fig 5), the heat supplied was 125J, use the first law to find the gain in internal energy (ΔU) of the gas.

We have $Q = \Delta U + W$, with $Q = 125J$, and $W = 50\ J$.
The gain in internal energy $\Delta U = (125J - 50J) = 75J$

Practice Questions

1. A quantity of gas has a temperature of 27°C and loses 10% of its kinetic energy. Calculate its initial and final kelvin temperature.

2. The diagram below shows an adiabatic expansion.....no heat gain or loss.

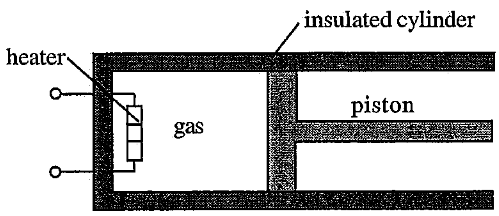


This time 300J of work are done. Because the gas is *expanded* we must take the work done to be *positive.*
Calculate:
(a) the change in internal energy.
(b) has the temperature gone up or down?

3. A gas is heated at a constant pressure of 10^6 Pa and its volume increases by 8 litres. Calculate the work done by the gas as it expands. If the heat energy supplied to the gas is 10kJ, calculate the gain in internal energy of the gas.

4. An engine discharges $15m^3$ of hot gas at a pressure of 1.0×10^5 Pa to the atmosphere that is also at a pressure of 1.0×10^5 Pa. The gas then cools and contracts to one third of its original volume.
(a) Calculate the work done on the exhaust gas.
(b) During the cooling process, 5.0 MJ of heat is transferred to the atmosphere. Calculate the change in internal energy of the gas.

5. The diagram shows an insulated cylinder fitted with a perfectly fitting frictionless piston. The cylinder contains a fixed mass of gas and a heater.



Two experiments are performed:

Experiment 1
The heater provides 150 J of energy with the piston held in a fixed position. The temperature rise of the gas is 30K.

Experiment 2
The heater again supplies 150J of energy with the piston free to move so that the gas expands at constant pressure and does some useful work W. In this case, the temperature rise is 18K and the efficiency is 40%

(a) Explain why there is a difference in the temperature rise in each case.
In the first experiment, calculate
(b) the gain in internal energy of the gas
In the second experiment, calculate
(c) the work done by the expanding gas
(d) the gain in internal energy of the gas
In the second case, the gas expanded at a constant pressure of 1.0×10^5 Pa, calculate
(e) the change in volume of the gas.

Answers

1. The kinetic energy (internal energy) $\propto$ kelvin temperature.
If 10% of kinetic energy lost then 10% of kelvin temperature 'lost'.
27°C = 300K. 10% loss gives 270K (270-273)°C = -3°C

2. In all cases, Q = ΔU + W.
For adiabatic change, Q = 0. Gas expands so work done is Positive.
0 = ΔU + 300J
ΔU = -300J. That is a decrease in internal energy with a corresponding decrease in temperature.

3. $P = 10^6$ Pa, $\Delta V = 8\,l = 8\times10^{-3}\,m^3$.
Work Done $= P \times \Delta V$
$= 10^6 \times 8\times10^{-3} = 8\times10^3$J (8kJ)
Q = ΔU + W
10kJ = ΔU + 8kJ $\Rightarrow$ ΔU = 2kJ (the increase in internal energy)

4. (a) Work done $= P\times\Delta V$
$= 1.0\times10^5(15-5)$
$= 1.0\times10^6$ J
This is a compression so $W = -1.0\times10^6$J

(b) The exhaust gas is the system, it loses 5.0×10^6J and so, $Q = -5.0\times10^6$J
First Law, Q = ΔU + W
$-5.0\times10^6 = \Delta U + (-1.0\times10^6)$
$-4\times10^6 = \Delta U$
i.e., a loss in internal energy of 4MJ

5. (a) In the first part the piston is fixed so no work is done. All the 150J increases the internal energy and consequently the temperature. In the second part the piston is free to move, work is done, so now less energy is available to increase the internal energy with a corresponding lower temperature rise.

(b) Q = ΔU + W, because W = 0, Q = ΔU = 150J
All the energy supplied (150J) increases the internal energy

(c) efficiency = work / energy supplied
40/100 = W/150. $\Rightarrow$ W = 60J

(d) Q = ΔU + W $\Rightarrow$ 150J = ΔU + 60J $\Rightarrow$ ΔU = 90J

(e) $W = P \times \Delta V$
$60 = 1.0\times10^5 \times \Delta V$
$\Delta V = 60 \times 10^{-5}\,m^3$ or 0.6 litres

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Physics Factsheet

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Mass Defect and Mass Loss

Mass Defect

The mass of a nucleus is less** than the combined mass of its protons and neutrons (nucleons). The missing mass is called the mass defect.

***A hydrogen nucleus consisting of one proton is the only exception to this rule*

> ***Worked Example 1a***
> ***A lithium Li-7 nucleus has a mass of 7.01436 u.***
> ***What is its mass defect? Use the fact that a proton has a mass of 1.00728 u and a neutron 1.00867u. (1u= 1/12 th mass of a carbon atom = 1.661×10^{-27} kg)***
>
> ***Answer***
> *The reaction can be summarised by the equation:*
>
> $$3\,{}^{1}_{1}p + 4\,{}^{1}_{0}n \rightarrow {}^{7}_{3}Li$$
>
> *Mass of reactants = 3×1.00728 u + 4×1.00867u = 7.05652u*
> *Mass of product = 7.01436 u*
> *Mass Defect = 0.042 16u*
> *or 7.00278×10^{-29} kg*

> *Mass Defect = Mass of all reactants – Mass of all products*

Binding energy

Protons and neutrons also lose potential energy (P.E.) when they are bound together in a nucleus. The difference in the P.E. is equal to the binding energy of the nucleus.

> *Binding energy = energy needed to tear a nucleus apart into individual protons and neutrons.*

Mass-energy equivalence

Einstein suggested that the binding energy of a nucleus is the mass defect of the nucleus. His most famous equation gives the energy equivalent of mass:

$\Delta E = \Delta mc^2$ where energy E is measured in joules when mass m is measured in kg and the speed of light c is a constant 3×10^8 m/s. So binding energy = mass defect × c^2 .

> ***Worked Example 1b***
> ***Find the binding energy of the Li-7 nucleus.***
>
> ***Answer***
> *$\Delta E = 7.00278 \times 10^{-29}$ kg $\times (3.00 \times 10^8$ m/s$)^2 = 6.30 \times 10^{-12}$ J*
> *Small amounts of energy can also be expressed in MeV (megaelectonvolt), where*
> *1 MeV = 1.6×10^{-13} J*
> *The binding energy of the nucleus is 6.3×10^{-12} J or 39.4 MeV*

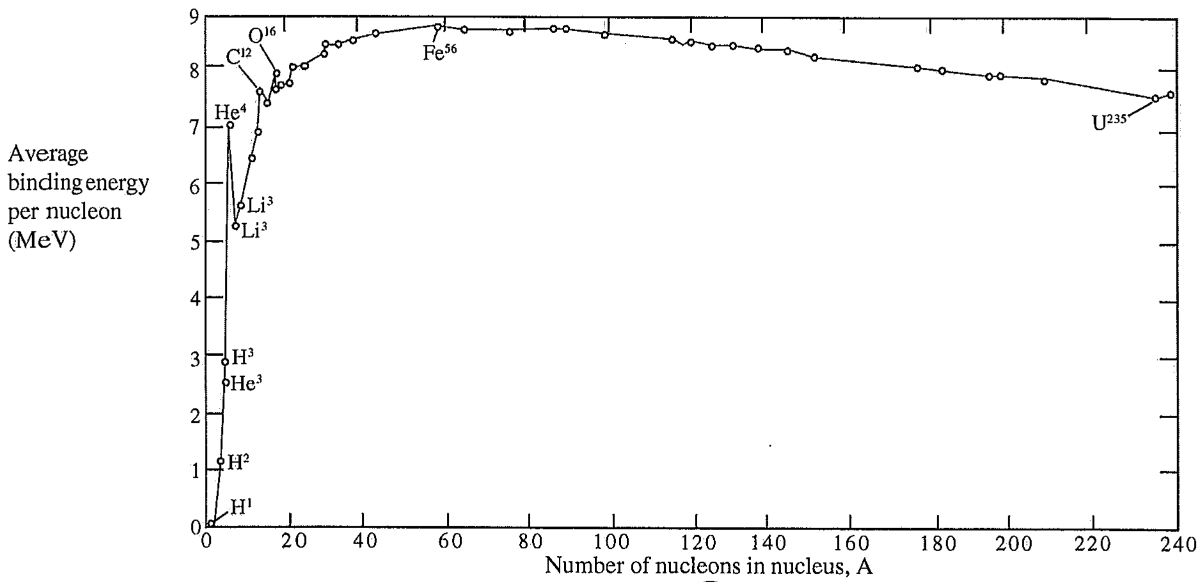
> ***Exam Hint:** Mass and energy are equivalent, so although mass may not appear to be conserved in a particular interaction, the mass-energy of the system stays constant.*

Binding energy per nucleon

This is found by dividing the binding energy of the nucleus by the number of nucleons. Fig 1 shows how binding energy per nucleon changes as nucleon number increases.

* binding energy/nucleon increases with nucleon number for relatively small nuclei
* the most stable nuclei have the highest binding energy per nucleon
* binding energy / nucleon decreases with nucleon number for relatively large nuclei

Fig 1. Binding energy per nucleon



Binding energy per nucleon and stability

The nucleus is held together by the attractive 'strong nuclear force' which acts over short distances. For low numbers of nucleons the strong force increases with nucleon number (positive slope left hand side of graph). Tightly held nucleons means the nucleus is more stable. When nucleon numbers exceed about 60 the nucleons cannot be packed together close enough for the strong nuclear force to hold all the nucleons as tightly, so stability now starts decreasing with nucleon number (negative slope right hand side of graph).

The higher the binding energy per nucleon, the more stable the atom.

Worked Example 1c
Find the binding energy per nucleon for Li-7

Answer
Binding energy per nucleon for Li-7
= Binding energy / number of nucleons
= 39.4 MeV / 7 = 5.6 MeV

Unstable atoms

An unstable nucleus breaks up relatively easily. This can happen spontaneously as in **radioactive decay** or can be forced as in a **fission** reaction.

Radioactive Decay occurs when an unstable nucleus emits α, β or γ radiation to form smaller 'daughter' nuclei. Daughter nuclei are always more stable because their binding energy per nucleon is higher than the 'parent' nuclei.

Worked Example 2
Here is an example of alpha decay:

$$^{228}_{90}Th \rightarrow {}^{224}_{88}Ra + {}^{4}_{2}\alpha$$

Given the following masses:
Mass of Th- 228 = 227.97929 u
Mass of Ra -224 = 223.97189 u
Mass of α = 4.00151 u

(i) Show that the binding energy / nucleon of thorium is greater than that for radium.
(ii) Find the mass defect of the reaction.
(iii)Find the energy equivalent of the mass defect and suggest where the energy appears.

Answers
(i) Binding energy per nucleon for Th-228
Binding energy = mass defect (kg) × c² (m/s)²
$= 1.661\times10^{-27}((90\times1.007276 + 138\times1.008665) - 227.97929) \times (3\times10^8)^2$
$= 2.799 \times 10^{-10}$ J
Binding energy / nucleon $= 2.799 \times 10^{-10}$ J /228
$= 1.223 \times 10^{-12}$ J/nucleon

Binding energy per nucleon for Ra-224
$= 1.661\times10^{-27}((88\times1.00728 + 136\times1.00867) - 223.97189)$ J $\times (3\times10^8)^2/224$
$= 1.233 \times 10^{-12}$ J / nucleon

(ii)Mass defect = mass left hand side of equation – right hand side of reaction
= 0.00589 u
(iii) Energy equivalent = mass defect $\times c^2 = 8.80 \times 10^{-13}$ *J or 5.5 MeV*
The missing mass appears as the kinetic energy of the particles after the decay.

Exam Hint: *When a decaying nucleus becomes 'lighter', you could say that the nucleus has lost mass, but this is not the mass defect. If you can show that the **total** mass of the product nuclei is less than the reactants, then this difference is the **mass defect.***

Fission

Fission means splitting up. It is possible to fission large unstable nuclei by bombarding them with neutrons, producing 2 nuclei which are more stable. This is the equation of a fission reaction which takes place in nuclear power stations where the energy released is used to generate electricity.

$$^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{241}_{26}Ba + {}^{92}_{36}Kr + 3\,{}^{1}_{0}n$$

Exam Hint: *Fission reactions refer to the right hand slope of the binding energy per nucleon / nucleon number graph. Daughter nuclei are to the left of the parent nucleus because daughter nuclei are smaller and more stable than the parent nuclei.*

Other types of mass-energy interactions

Fusion

Fusion means joining together. Energy is generated in stars such as the Sun by fusion of hydrogen nuclei into helium nuclei. Here is an example of a fusion reaction.

$$^{2}_{1}H + {}^{3}_{1}H \rightarrow {}^{4}_{2}He + {}^{1}_{0}n$$

Energy released by this reaction is 16.875 MeV
Attempts to discover how fusion reactions could be a source of energy for power stations have been made over the past twenty years, but so far without success.

Exam Hint: *Fusion reactions refer to the left hand side of the binding energy per nucleon/nucleon number graph. Products of fusion reactions are larger and more stable than parent nuclei so are found to the right of the parent nuclei on the graph.*

Annihilation

When a particle meets its antiparticle there is instant annihilation of mass. The mass-energy of the pair appears as new particles and or radiation. For example when an electron reacts with its antiparticle (positron) 2 photons are produced which radiate in opposite directions so that momentum is conserved.

Creating particles with mass

The opposite reaction to annihilation occurs when a photon materialises into an electron and positron. A photon's energy is related to its frequency by the equation:

$E = h \times f$ where E is energy in joules
h is Planck's constant 6.63×10^{-34} Js
f is frequency in hertz

Worked Example 3

If an electron has a mass 5.5 × 10⁻⁴ u, what is the minimum energy of a photon needed to create an electron / positron pair?

Solution

Energy equivalent of electron mass (J) = mass of electron (kg) × $c^2 (m/s)^2 = 2 \times 5.5 \times 10^{-4} \times 1.661 \times 10^{-27} \times (3.00 \times 10^{8})^2 = 1.6 \times 10^{-13}$ J

Related frequency = energy equivalent / Planck's constant

$f = 1.6 \times 10^{-13}$ J / 6.63×10^{-34} Js = 2.5×10^{20} Hz (frequency in gamma wave range)

Practice Questions

Use the following data where necessary (1MeV = 1.602×10^{-13} J , 1u= 1.661×10^{-27} kg)

1. The output power of a nuclear reactor is provided by nuclear fuel which decreases in mass at a rate of 4×10^{-6} kg per hour. What is the maximum power of the reactor?

2. (a) What type of reaction is shown by the equation? Give an example of where it could take place. $^{235}_{92}U \rightarrow {}^{241}_{26}Ba + {}^{92}_{36}Kr + 2\,{}^{1}_{0}n$

   (b) Calculate the binding energy per nucleus of the parent and daughter nuclei given the following data: proton mass =1.00728 u , neutron mass=1.00867u

   (c) Comment on your results in part b.

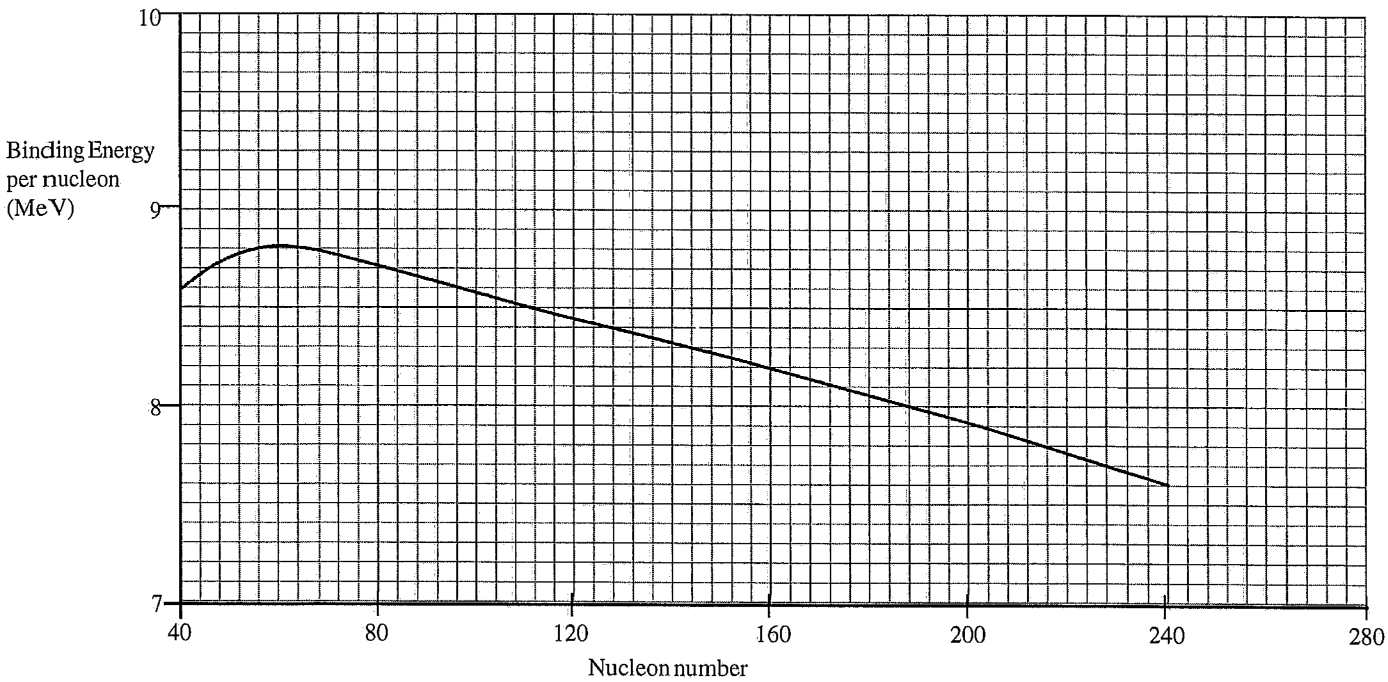
3. The energy released in this fusion reaction: $^{3}_{1}H + {}^{2}_{1}H \rightarrow {}^{4}_{2}He + {}^{1}_{0}n$ is 16.875 MeV

   Find the mass of $^{3}_{1}H$ given the following data:

   H-2 nucleus =3.343×10^{-27} kg, He-4 nucleus = 6.645×10^{-27} kg, neutron = 1.675×10^{-27} kg

4 Explain what is meant by 'decays spontaneously' and how consideration of the masses of particles involved in a proposed decay helps in deciding whether the decay is possible.

5 Part of a graph of how binding energy per nucleon varies with nucleon number is shown:



   (a) State the value of the nucleon number for the nuclides that are most stable, giving a reason.

   (b) Assuming that the fission of U-235 gives 2 daughter nuclei of roughly equal nucleon numbers and 200 MeV of energy, use the graph to justify the figure explaining your reasoning.

Answers

1. Find the energy equivalent of the mass used per hour. Then divide by 3600s to find the energy produced per second (power in watts).
   $\Delta E = \Delta mc^2 = 3.6 \times 10^{11}$ J
   Power = 3.6×10^{11} J / 3600s= 100 MW

2 (a) Fission reaction, takes place in reactors of nuclear power stations

(b) Binding energy: U-235 = 92×1.00728 u + 143×1.00867u = 236.90957u
Ba-141 = 56×1.00728 u + 85×1.00867u =142.14463 u
Kr -92 = 36×1.00728 u + 56×1.00867u = 92.74760 u

(c) The combined mass of the daughter nuclei (234.89223 u) is less than the mass of the parent (mass defect). Hence energy will be generated by this fission reaction.

3. Find the mass equivalent of 16.875 MeV.
   $\Delta m = \Delta E / c^2 = 16.875 \text{ MeV} \times 1.602 \times 10^{-13} \text{ J} / 9 \times 10^{16} \text{ m}^2/\text{s}^2 = 3.004 \times 10^{-29}$ kg
   Mass of reactants = mass of products + mass defect
   Mass of H-3 = Mass of products + mass defect – mass of H-2 = 5.007×10^{-27}kg.

4. Spontaneous decay means nuclear disintegration without any external stimulus.
   Only possible if the mass of the products of disintegration is less than the mass of the decaying nucleus.

5. (a) Nuclei with higher binding energy per nucleon are more stable.
   The graph shows the highest binding energy per nucleon at nucleon numbers of around 54-64.

   (b) Graph shows binding energy per nucleon for nuclei with 240 nucleons is approximately 7.6 MeV whereas for nuclei with 120 nucleons the level is around 8.4 MeV.
   The difference in binding energy per nucleon: 8.4 MeV - 7.6 MeV = 0.8 MeV.
   Each nucleus of 240 nucleons gives up 240×0.8 MeV = 192 MeV (200 MeV to 1 significant figure)

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Physics Factsheet

www.curriculumpress.co.uk Number 58

Nuclear Reactors

This Factsheet explains the process by which energy is released by nuclear fission and how this energy is harnessed by a nuclear reactor.

Nuclear Fission

Nuclear fission is the splitting of an unstable, larger nucleus into two smaller nuclei. In a nuclear reactor the larger nucleus, which is usually an isotope of uranium, is unstable because it has too many neutrons. The uranium has been made unstable as it has been forced to absorb an additional neutron. The uranium nucleus has become unstable and split after absorbing a neutron. This is called **neutron-induced fission**.

An example of a fission reaction would be:

$$^{235}_{92}U + ^{1}_{0}n \rightarrow ^{141}_{56}Ba + ^{92}_{36}Kr + 3^{1}_{0}n$$

This nuclear transformation equation shows the uranium nuclei and the neutron on the left hand side of the arrow. These are the reactants. In this example, barium and krypton have been produced along with 3 additional neutrons. These products are shown on the right hand side of the arrow.

Note how the total of the atomic number (the number written as subscript) on both sides of the equation is the same, 92 + 0 = 92 and 56 + 36 + 0 = 92 on the right hand side. The atomic number represents the number of protons in each nucleus. No protons have been created or destroyed so the total number of protons remains the same.

Also, the total of the nucleon numbers (the number written as superscript) on both sides of the equation is the same, 235 +1 = 236 on the left hand side and 141 + 92 + (3×1) = 236 on the right hand side. The nucleon number represents the total number of protons plus neutrons in each nucleus. No protons or neutrons are destroyed during fission so the total number of protons plus neutrons remains the same.

Nuclear Fission
Nuclear fission is the splitting of a large, unstable nucleus into two, smaller, more stable nuclei. Neutron-induced fission is the fission of a large nucleus that has become unstable due to the absorption of an additional neutron.

Nuclear Transformation Equations.
In any nuclear transformation equation, such as the example shown above for the fission of uranium, the total nucleon number and the total atomic number on both sides of the equation must be the same.

How is energy released from nuclear fission?

When the total mass of all of the products in a fission reaction is calculated, it is less than the total mass of all of the reactants. In other words the products have a smaller mass than the reactants. The reaction produces a loss of mass called a **mass defect**. This mass defect has occurred due to a release of energy by the reaction, according to the equation:

$E = mc^2$ where E = *energy released (J)*
m = *mass defect (kg)*
c = *speed of light in a vacuum* = 3.0×10^8 ms^{-1}.

This energy is released in the form of heat energy. This means that the product nuclei and neutrons move away from the reaction with incredible speeds as they have a lot of kinetic energy.

Consider the masses of the nuclei from the fission example given above:
Mass of uranium nucleus = 235.04 atomic mass units
Mass of a neutron = 1.01 atomic mass units.

So the total mass of reactants = 235.04 + 1.01
= 236.05 atomic mass units.

Mass of barium nucleus = 140.91 atomic mass units
Mass of krypton nucleus = 91.91 atomic mass units
Mass of 3 neutrons = 3 × 1.01 = 3.03 atomic mass units.

So the total mass of products = 140.91 + 91.91 + 3.03
= 235.85 atomic mass units.

Mass defect = mass of products – mass of reactants
= 236.05 – 235.85 = 0.20 atomic mass units

Now, one atomic mass unit is equal to 1.6605×10^{-27} kg. This means that the mass defect from our fission example is:

Mass defect = $0.20 \times 1.6605 = 0.3321 \times 10^{-27}$ kg

The energy released from this fission can now be calculated:

Energy released,
E = mass defect × c^2
$E = (0.3321 \times 10^{-27}) \times (3 \times 10^8)^2 = 2.99 \times 10^{-11}$ J

This amount of energy may not seem like a huge amount but you must remember that this is for one reaction only and in a nuclear reactor there will be a huge number of reactions occurring in a short space of time, leading to a huge amount of energy being released.

Energy Release From Fission
When fission of a large nucleus occurs the products have a smaller mass than the reactants. This apparent loss of mass or mass defect is released as heat energy, giving the reactants large amounts of kinetic energy.

The amount of energy released is related to the mass defect by the equation:

$E = mc^2$ *where:* E = *energy released (J)*
m = *mass defect (kg)*
c = *speed of light in a vacuum* = 3.0×10^8 *m/s.*

Exam Hint: *When doing this sort of calculation, check you are using the correct units - kg for mass, not atomic mass units*

Typical Exam Question
The nuclear transformation equation below represents the fission of uranium.

$$^{235}_{92}U + ^{1}_{0}n \rightarrow ^{146}_{57}La + ^{87}_{X}Br + 3^{1}_{0}n$$

(a) What is the missing atomic number of bromine? [1]
(b) Given the following masses of the nuclei involved in the fission, calculate the mass defect, in kg, caused by the fission. [4]

Mass of bromine nucleus = 86.92u
Mass of neutron = 1.01u
Mass of lanthanum nucleus = 145.90u
Mass of uranium nucleus = 235.04u
1u = 1.6605×10^{-27} kg

(c) Calculate the energy released in joules from one fission of uranium through this transformation. [2]

Answers
(a) *In a fission reaction the total atomic number must be equal before and after the fission. Before the fission the total atomic number is 92+0 = 92. The total number after the fission must also be 92. Missing nucleon number = 92 – 57 = 35* ✓

(b) *The mass defect is found by calculating the difference between the masses of all of the products and all of the reactants. To turn the mass defect into kilograms the conversion factor that is given in the question is used.*
Mass of reactants = uranium mass + neutron mass
= 235.04 + 1.01 = 236.05u ✓

mass of products = bromine mass + lanthanum mass + mass neutrons
= 86.92 + 145.90 +3(1.01) = 235.85 u ✓

Mass defect = mass of reactants – mass of products
= 236.05 – 235.85 = 0.20 u ✓

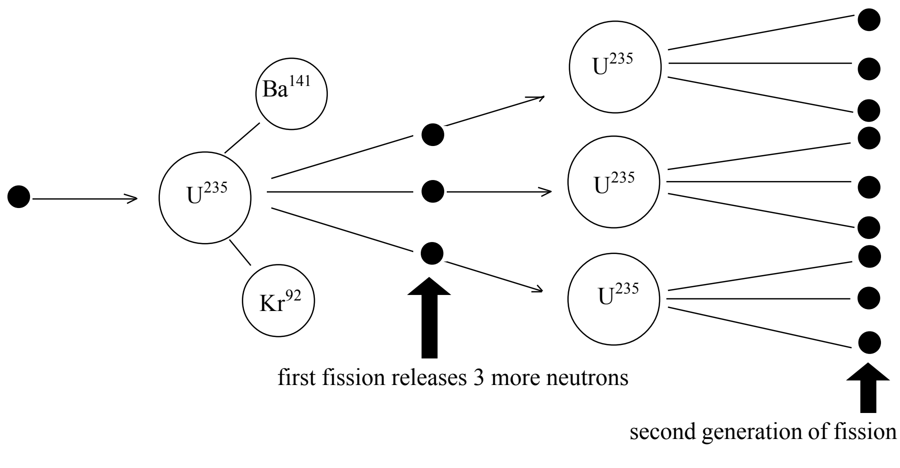
Mass defect = $(0.20)(1.6605 \times 10^{-27}) = 3.21 \times 10^{-28}$ *kg* ✓

(c) *The energy released is calculated by using* $E = mc^2$ ✓
$E = mc^2 = (3.21 \times 10^{-28})(3 \times 10^8)^2 = 2.89 \times 10^{-11}$ *J* ✓

Chain Reactions

It has already been explained that when a neutron-induced fission occurs a heavy unstable nucleus splits into two smaller nuclei. Usually, the fission of a uranium nucleus also produces two or three additional neutrons. Each one of these additional neutrons is now able to collide with a nearby uranium nucleus and cause a further fission. In turn each of these fission reactions will liberate further neutrons that are able to cause further fission reactions. The process can be self-sustaining by forming a **chain reaction**.

The diagram below represents the first few stages in a chain reaction:



One fission may release 3 neutrons. These 3 neutrons will go on to cause 3 further fissions releasing 9 neutrons. These 9 neutrons will go on to cause 9 fissions releasing 27 neutrons, etc, etc.

Each time a fission reaction occurs there is a mass defect and a release of energy. The process of fission takes a very short time, about 0.001 seconds. This means that within a very short space of time there are a huge number of fission reactions taking place, releasing a fantastic amount of energy.

Self Sustaining Chain Reactions

Not all of the neutrons that are produced by a fission reaction will go on to cause further fission of other uranium nuclei. Some neutrons will escape the uranium into the surroundings.

If, on average, one neutron from each fission reaction goes on to cause a further fission then the chain reaction will be just self-sustaining at a steady rate and the reaction is called critical.

If, on average, more than one neutron from each fission reaction goes on to cause a further fission then the chain reaction will very quickly spiral out of control and become explosive. This type of reaction is called super-critical and forms the basis of an atomic bomb!

If, on average, less than one neutron from each fission reaction goes on to cause a further fission then the chain reaction will die out.

The number of neutrons that escape the uranium and do not cause further fission reactions is dependent on two factors:

(i) The mass of uranium. A smaller mass has a larger surface area in proportion to its volume and so there is a greater chance of a neutron escaping the uranium with a smaller mass. The mass of a material required to produce a self sustaining critical reaction is called the **critical mass.**

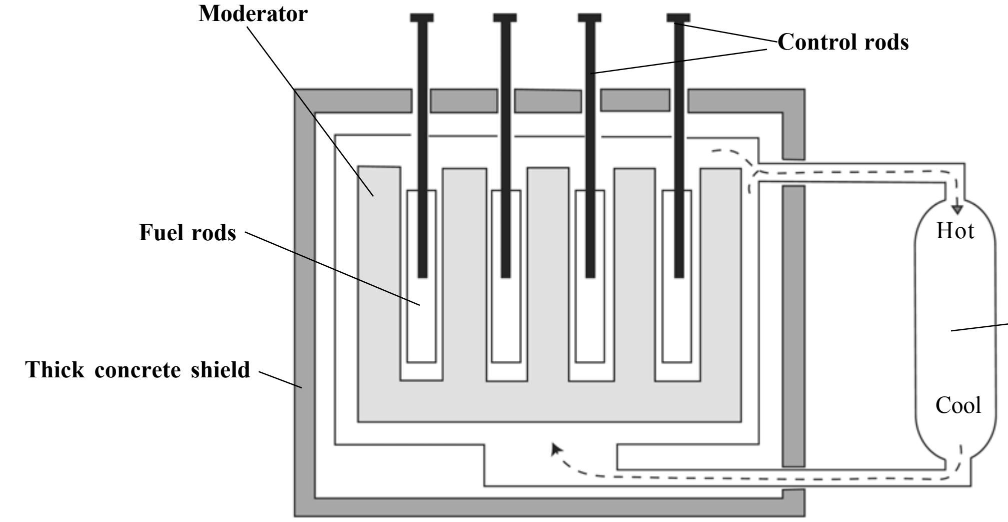
(ii) The shape of the uranium. A long thin piece of uranium will allow a lot of neutrons to escape but a sphere will keep more neutrons inside the uranium for longer, giving them more chance to cause further fission reactions. This means that the critical mass of a material depends on the shape of the material.

The critical mass for uranium is about 15kg if the uranium is in the shape of a sphere. This is a sphere with a radius of about 6cm or the size of a grapefruit. Slightly more uranium in the sphere would produce a super critical chain reaction.

Critical Mass
The critical mass is the mass of material required to create a chain reaction that is self sustaining at a steady rate. This is called a critical chain reaction and is created when one neutron, on average, from every fission goes on to cause a further fission of a nucleus.

Nuclear Fission Reactors

The diagram below represents a simplified design of a gas cooled nuclear reactor:



The important parts and components of a nuclear reactor are labelled on the diagram. The function of these components are explained below

Moderator - In naturally occurring uranium only about 1 atom in 140 is a uranium 235 ($^{235}_{92}U$)isotope. The rest of the atoms are uranium 238 ($^{238}_{92}U$).

Uranium 235 requires comparatively slower moving neutrons to cause fission. Uranium 238 requires comparatively faster moving neutrons to cause fission.

The average speed of naturally occurring neutrons from a fission reaction is usually somewhere in between these two speeds. Therefore, in a nuclear reactor a moderator is used. This slows neutrons down to a speed that is more likely to cause fission in uranium 235 nuclei.

The neutrons collide with the nuclei of the moderator and transfer kinetic energy to the nuclei. Neutrons that have been slowed down by a moderator are called ***thermal neutrons***. Most reactors use graphite or water as the moderator.

Control Rods - These are able to absorb neutrons from the reactor. Neutrons that are absorbed by the control rods do not go on to cause further fission reactions. The control rods can be moved in and out of the fuel to absorb more or less neutrons. In this way the reactor can support a self-sustaining reaction at a steady rate where only one neutron from each fission reaction goes on to cause a further fission reaction.

If the nuclear reactor is producing too much energy and is in danger of becoming super-critical the control rods are lowered further into the reactor and they absorb more neutrons. If the reactor is not producing enough energy and is in danger of dying out then the control rods are lifted further out of the reactor and less neutrons are absorbed.

Cadmium and boron are good materials to use as control rods as they are very good at absorbing neutrons.

Coolant - The coolant that is labelled in the diagram above is a gas; carbon dioxide is usually used, but water is also commonly used. The coolant passes through the reactor and becomes hot. The hot coolant then passes through a heat exchanger. Inside the heat exchanger the coolant transfers its heat to steam. The steam is then used to turn a turbine and drive a generator. The coolant continually circulates around the reactor core, taking away the heat energy that is produced so that it can be converted into electrical energy.

Fuel Rods - The most commonly used fuel is 'enriched uranium'. This is uranium that has had the proportion of uranium 235 isotope increased to around 3%. The fuel rods are long and thin to facilitate the easy escape of neutrons into the moderator. The moderator slows the neutrons down before they return to the fuel rod and cause another fission reaction.

Thick Concrete Shield - This acts as a barrier to stop any neutrons escaping from the reactor core. The thickness of the concrete is typically as much as 5 metres. This thickness of concrete will stop any neutrons, beta particles and gamma photons, which are all produced in the core, from escaping. Neutrinos can also be produced inside the core and these will pass through the concrete but they are not harmful.

Choice of Moderator

The choice of material to use as a moderator is crucial in a nuclear reactor. An effective moderator must slow down the neutrons quickly without absorbing them. The kinetic energy of the neutron is reduced through collisions with the nuclei of the moderator. If the moderator nuclei are too heavy, the neutrons will simply bounce off and lose little kinetic energy. If the moderator nuclei are stationary and equal to the mass of the neutron then the neutron will become stationary after the collision. An effective moderator should therefore nave nuclei that are moving and have about the same mass as a neutron.

Candidate 1 – Hydrogen Gas : Although the nuclei (single protons) are moving and have the same mass as a neutron it is impractical to use a gas as a moderator inside the reactor core.

Candidate 2 – The hydrogen nuclei within water molecules : The hydrogen nuclei are moving and have the same mass as the neutrons. Unfortunately hydrogen nuclei are prone to absorbing neutrons to become the hydrogen isotope – deuterium.

Candidate 3 – The deuterium nuclei within 'heavy water' molecules: Water molecules can also be formed that use deuterium atoms instead of hydrogen atoms. These nuclei are only twice as heavy as the neutrons and they are moving. The draw back is that 'heavy water' is inconvenient and expensive to produce in a pure form.

Candidate 4 – Carbon Graphite: Carbon nuclei have about 12 times the mass of a single neutron and so they are less effective at slowing neutrons down. The neutrons have to undergo many collisions with the carbon graphite nuclei before they are slow enough to cause fission in a uranium 235 nucleus. Despite this drawback the carbon graphite is cheap and easily obtained. It is for this reason that carbon graphite is used in nuclear reactors as the moderator.

Characteristics of an effective moderator material

An effective moderator material must slow down a fast moving neutron to a speed that is capable of causing fission in a uranium 235 nucleus through as few a collisions as possible with the nuclei of the moderator material. This means that an effective moderator nucleus should be moving and have about the same mass as a neutron.

Typical Exam Question
The *control rods*, the *coolant* and the *thick shield* surrounding the reactor core are essential parts of a fission nuclear reactor.
For each of these components:
(i) explain its function
(ii) suggest one suitable material
(iii) give one essential physical property that the material must have. [12]

Answers
Control Rods
(i) *Control rods control the rate of reaction so that the chain reaction is just self sustaining or critical.*✓
The control rods do this by absorbing neutrons from the reactor core so that, on average, one neutron from each fission reaction goes on to cause a further fission. To achieve a critical reaction the control rods can be✓ *lowered and raised into the reactor core to absorb more or less neutrons when required.*
(ii) *Cadmium is a suitable material.* ✓
(iii) *The control rod material must be a good absorber of neutrons.*✓

Coolant
(i) *The coolant transfers heat energy away from the reactor core.*✓
The coolant is taken to a heat exchanger where it is used to heat up ✓ *steam that will eventually turn a turbine and drive an electricity generator.*
(ii) *Water, or carbon dioxide gas.*✓
(iii) *The coolant must have a high heat capacity to transfer as much heat energy as possible.* ✓

Thick shield surrounding core
(i) *The thick shield prevents harmful radiation from escaping the reactor core.*✓
The shield absorbs neutrons, beta particles and gamma photons that are produced inside the core. ✓
(ii) *Concrete is a suitable material.*✓
(iii) *The thick shield must be a good absorber of radiation.*✓

Qualitative (Concept) Test
1. What is nuclear fission?
2. Why is energy released during the nuclear fission process and in what form is the energy released?
3. (a) With reference to the neutrons involved in the nuclear fission process, under what circumstances would a chain reaction be just self sustaining?
   (b) Under what circumstances would a chain reaction become 'super-critical?
4. What two physical factors affect the critical mass of material that can undergo neutron-induced fission? Explain how each factor effects the critical mass.
5. In a nuclear reactor what are the functions of the following components:
   (a) The moderator
   (b) The coolant
   (c) The control rods
6. What are the characteristics needed by a good moderator, and why?

Quantitative (Calculation) Test
1. When $^{235}_{92}U$ is bombarded by neutrons, two possible fission products are $^{95}_{39}Y$ and $^{139}_{53}I$.
   (a) Give a nuclear transformation equation for this process.
   (b) Given the following masses of the nuclei involved in the fission, calculate the mass defect of the fission.
   Mass of uranium 235 = 235.04u Mass of yttrium 95 = 94.91u
   Mass of iodine 139 = 138.91u Mass of one neutron = 1.01u
   $1u = 1.6605 \times 10^{-27}$ kg
   (c) What is the energy, in joules, that is released during one of the fission reactions?

Exam Workshop
This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's mark scheme is given below.

Natural Uranium consists of 99.3% $^{238}_{92}U$ and 0.7% $^{235}_{92}U$. In many nuclear reactors, the fuel consists of enriched uranium.

(a) (i) Explain what is meant by enriched uranium. [1]
Uranium with more fuel in it. *0/1*

This answer needs to be more specific. The fuel of a nuclear reactor is uranium 235. The proportion of uranium 235 is around about 3% in enriched uranium.

(ii) Why is enriched uranium used rather than natural uranium in nuclear reactors? [2]
There is more fuel in enriched uranium. *0/2*

Again, it is specifics that are lacking in this answer. No mention has been made of fission and how fission of uranium 235 is the source of energy release in the reactor.

(b) (i) Explain how the rate of heat production is controlled in a nuclear reactor. Your answer should refer to the thermal neutrons involved in the fission process. [3]
The heat given out by the reactor increases if the control rods are removed as there are more neutrons. When the control rods are inserted into the reactor there are fewer neutrons and the heat given out decreases. *1/3*

This answer is insufficiently detailed for this number of marks - the candidate should include more information about the role of neutrons in fission as is indicated in the question.

(ii) Explain why the nuclear fuel in a nuclear reactor is shaped as many long thin rods. [4]
The neutrons must leave the fuel so that they can be slowed down *1/4*

Again, what the candidate has mentioned is correct but there needs to be more detail in a question with this many marks.

Examiner's answers
(a) (i) The quantity of uranium 235 is greater in enriched uranium than in naturally occurring uranium.✓
(ii) The fission reactions occur in uranium 235 and so the amount of available fuel is greater in enriched uranium.✓
(b) (i) Each fission reaction releases 2 or 3 neutrons.✓
Only one neutron from every fission reaction is required to cause a further fission reaction.✓
The control rods absorb any neutrons that are not needed to ensure that the chain reaction remains at a steady rate.✓
(ii) The neutrons must be slowed down by the moderator.✓
The neutrons must therefore escape the fuel rods.✓
Neutrons can more easily escape a long thin rod than any other shape.✓
Also, the long thin fuel rods are more easily replaced, in stages, than a larger piece of fuel.✓

Quantitative Test Answers

1. (a) $^{235}_{92}U + ^{1}_{0}n \rightarrow ^{139}_{53}I + ^{95}_{39}Y + 2^{1}_{0}n$ ✓✓
   (b) Mass of reactants = uranium mass + neutron mass
   = 235.04 + 1.01 = 236.05u✓
   Mass of products = yttrium mass + iodine mass + mass neutrons
   = 94.91 + 138.91 +2(1.01) = 235.84 u✓
   Mass defect = mass of products – mass of reactants
   = 236.05 – 235.84 = 0.21 u✓
   Mass defect = $(0.21)(1.6605 \times 10^{-27}) = 3.49 \times 10^{-28}$ kg ✓
   (c) The energy released is calculated by using $E = mc^2$.
   $E = mc^2 = (3.49 \times 10^{-28})(3 \times 10^8)^2 = 3.141 \times 10^{-11}$ J ✓✓

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Physics Factsheet

September 2001 Number 21

Nuclear Transformations & Binding Energy

This Factsheet will explain:
- **the concept of binding energy, and how it relates to nuclear stability;**
- **how to use mass-energy equivalence in nuclear processes;**
- **the principles involved in fission and fusion reactions;**

Before studying this Factsheet, you should be familiar with basic atomic structure and equations for nuclear processes including radioactivity (covered in Factsheet 11 – Radioactivity I).

Binding Energy
The protons and neutrons in the nucleus of an atom are held together by the **strong nuclear force** (Factsheet 14 – Particle Physics). So if we imagine splitting a nucleus up into its separate protons and neutrons, it would require **energy**, because we would need to overcome the strong nuclear force.

Since energy is conserved:

$$\text{energy of nucleus} + \begin{array}{c}\text{energy put in to}\\ \text{separate nucleons}\end{array} = \begin{array}{c}\text{total energy of}\\ \text{separated nucleons}\end{array}$$

Since energy is being provided, the separated protons and neutrons must have more energy in total than the original nucleus. So if the nucleus was formed from its constituent particles, energy would be released.

> ***Binding energy*** *is the energy released when a nucleus is formed from its constituent particles*

Units
The standard SI units for mass and energy – the kilogram and joule – are too large to use conveniently on an atomic scale. Instead, the **unified atomic mass unit (u)** and the **electronvolt (eV)** are used.

> *1 electronvolt (eV) is the energy transferred to a free electron when it is accelerated through a potential difference of one volt.*
>
> $$\textit{1 unified atomic mass unit (u)} = \frac{\textit{mass of carbon} - 12 \textit{ atom}}{12}$$

It is necessary to be able to convert these to SI units:

1 eV = charge on electron in coulombs $\times$ 1 volt
= 1.602×10^{-19} J

Since the mass in grammes of one carbon-12 atom is its atomic mass (12) divided by Avagadro's number ($N_A = 6.02 \times 10^{23}$):

$$1\text{ u} = \frac{12/6.02\times10^{23}}{12} = 1.66 \times 10^{-24}\text{ g} = 1.66 \times 10^{-27}\text{ kg}$$

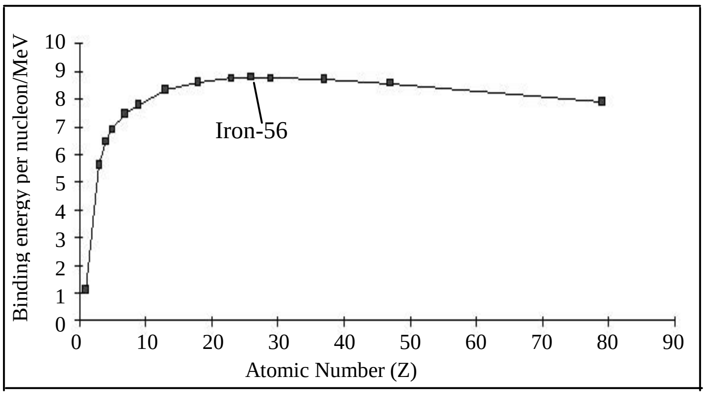
> ***Exam Hint: You need to know how to work out the relationship between electronvolts and joules, and between unified atomic mass units and kilograms, but you do not need to remember the actual figures.***

Binding energy per nucleon
Instead of looking at the total binding energy of a nucleus, it is often more useful to consider **binding energy per nucleon** – in other words, the total binding energy divided by the total number of nucleons.

For example, for the alpha particle, the total binding energy is 28.4 MeV Since there are four nucleons (two protons and two neutrons), the binding energy per nucleon is 28.4 ÷ 4 = 7.1 MeV.

The binding energy per nucleon gives an indication of the **stability** of the nucleus. A high binding energy per nucleon indicates a high degree of stability – it would require a lot of energy to take these nuclei apart.

Fig. 1 Binding energy per nucleon plotted against atomic no.



As figure 1 shows, the binding energy per nucleon varies considerably between nuclei.
- Nuclei near the peak of the curve are the most stable.
- The curve peaks at iron – 56.
- The graph of binding energy per nucleon against nucleon number is similar in form.

Nuclear processes and binding energy
Processes such as radioactive decay, fission and fusion involve the nucleons being rearranged into different nuclei. If the new nuclei produced have a **higher** binding energy per nucleon, then energy is **given out**. All **spontaneously occurring** processes involve energy being given out – i.e. the products have greater binding energy per nucleon than the original nuclei.

Radioactive decay is a spontaneous process. It always involves a less stable (i.e. lower binding energy per nucleon) nucleus decaying to form a more stable nucleus. Energy is therefore always given out in radioactive decay - in alpha-decay, for example, this energy is largely in the form of the kinetic energy of the alpha particle.

Nuclear fission involves a heavy nucleus (such as uranium) splitting to form two smaller nuclei and some neutrons. The nuclei produced will be nearer the peak of the graph – so energy is released. Fission can only occur with nuclei to the **right** of the peak.

Nuclear fusion involves small nuclei joining together to form a larger one – again, some neutrons are usually produced as well. The nucleus produced will always be nearer the peak of the graph – so again, energy is released. Fusion can only occur with nuclei to the **left** of the peak.

Mass-energy equivalence

Mass and energy are linked by Einstein's famous equation:

> $E = mc^2$,
> E = energy (J), m = mass (kg) and c = speed of light ($\approx 3 \times 10^8$ ms^{-1})

This equation is most often used in connection with a change in mass. It tells us, for example that a change in mass of 1kg is equivalent to a change in energy of $1 \times c^2 = 9 \times 10^{16}$ J.

Example: Calculate the energy change, in eV, equivalent to a mass change of 1u

1u = 1.66×10^{-27}*kg*
So energy change in J = $1.66 \times 10^{-27} \times c^2 = 1.49 \times 10^{-10}$ *J*
1 eV = 1.602×10^{-19} *J*

So energy change in eV = $\dfrac{1.49 \times 10^{-10}}{1.602 \times 10^{-19}} = 9.3 \times 10^8$*eV =930 MeV*

Mass defect

This relationship between mass and energy means that since a nucleus has less energy than its separated nucleons, the mass of the nucleus must be less than that of its constituent particles.

> *The **mass defect** of a nucleus is the difference between the mass of the nucleus, and the mass of its constituent particles.*
>
> *The mass defect is related to the binding energy by:*
> *Binding energy (J) = mass defect (kg)* $\times c^2$

All nuclei have a mass defect, apart from hydrogen – 1, whose nucleus consists of just a single proton.

The graph of mass defect per nucleon against atomic number is very similar to the graph of binding energy per nucleon against atomic number.

Example. The mass of an alpha particle is 4.00150u. The mass of a proton is 1.00728u, and the mass of a neutron is 1.00867u
(a) Calculate the mass defect of the alpha particle
(b) Calculate the binding energy of the alpha particle, giving your answer in MeV.

(a) Total mass of constituent particles = $2 \times (1.00728 + 1.00867)$
= 4.0319u
So mass defect = 4.0319 – 4.00150 = 0.0304u

(b) $0.0304\text{u} = 0.0304 \times 1.66 \times 10^{-27}\text{kg} = 5.0464 \times 10^{-29}$ *kg*
Binding energy (J) = $5.0464 \times 10^{-29} \times (3 \times 10^8)^2 = 4.542 \times 10^{-12}$*J*
Binding energy (eV) = $4.542 \times 10^{-12}/(1.602 \times 10^{-19}) = 2.84 \times 10^7$ *eV*
So binding energy = 28.4 MeV

The mass defect for a **reaction** may also be considered; this is the difference between the total mass of the products of the reaction and the total mass of the reactants. The equation $E = mc^2$ can be used to work out the energy released , once the mass defect is known.

Since spontaneous nuclear processes always involve energy being given out:

> **A spontaneous decay process always results in particles with a lower total mass**.

Calculation of the energy released in nuclear processes

To calculate the energy released in a process, the mass defect in kilograms should first be calculated, and then the equation $E = mc^2$ used to find the energy released.

***Example 1**: Calculate the energy released (in joules) in the process:*

$${}^{228}_{90}\text{Th} \rightarrow {}^{224}_{88}\text{Ra} + {}^{4}_{2}\text{He}$$

Masses: thorium-228= 228.02873u radium - 224 = 224.02020u
helium - 4 = 4.002603u

First we must find the mass defect:
Mass defect =(228.02873 – 224.02020 – 4.002603) u=5.927×10^{-3}*u*
Mass defect in kilograms = $5.927 \times 10^{-3} \times 1.66 \times 10^{-27}$
= 9.839×10^{-30} *kg*

So energy released = mass defect $\times c^2$
= $9.839 \times 10^{-30} \times (3 \times 10^8)^2 = 8.85 \times 10^{-13}$*J*

Typical Exam Question
The nuclear equation shown below represents the decay of radium to radon with the release of energy.

$${}^{226}_{88}\text{Ra} \rightarrow {}^{222}_{86}\text{Rn} + {}^{4}_{2}\text{He}$$

(a) Show, by appropriate calculations, that radon is more stable than radium. [5]
(b) Calculate the mass defect, in kg, for this reaction. [2]
(c) Calculate the energy released in this reaction. [2]
(d) Calculate the mass of radium (in kg) required to release 2000 MJ of energy (ignore energy released by decays of radon) [3]

In answering this question the following data may be useful
Masses: ***^{226}Ra = 226.02544u*** ***^{222}Rn = 222.01761u***
^{4}He= 4.002603u ***1u =*** 1.661×10^{-27}***kg***
neutron = 1.008605u ***proton = 1.007276u***
c = 3.0×10^{8}***ms***$^{-1}$ ***Avagadro's number*** $N_A = 6.02 \times 10^{23}$ ***mol***$^{-1}$

(a) To find the binding energy per nucleon of Ra:
Constituent nucleons: 88 protons + 138 neutrons
Mass of constituents = (88×1.007276)*u +* (138×1.008605)*u*
= 227.827778 ✓
Mass defect (mass of constituents – mass of nucleus):
= 227.827778u – 226.025u = 1.802778u ✓

To find the binding energy per nucleon of Rn:
Mass defect = $(86 \times 1.007276 + 136 \times 1.008605) - 222.01761$
= 1.778406u ✓

So forRa, mass defect per nucleon = 1.802778/226 = 7.977×10^{-3}*u*✓
for Rn, mass defect per nucleon = 1.778406/222 = 8.011×10^{-3}*u*

This tells us that the mass defect per nucleon (or binding energy per nucleon) is higher for Rn, so it is more stable ✓

(b) Mass defect =(226.02544– 222.01761–4.002604)u =5.227×10^{-3}*u* ✓
Mass defect (kg)=$5.227 \times 10^{-3} \times 1.661 \times 10^{-27} = 8.68 \times 10^{-30}$*kg* ✓

(c) Energy released = $mc^2 = 8.68 \times 10^{-30} \times (3.0 \times 10^8)^2$ ✓
= 7.8×10^{-13} *J*✓

(d) Number of atoms of Ra required = 2000 MJ/ 7.8×10^{-13} *J* ✓
= $2 \times 10^9 / 7.8 \times 10^{-13} = 2.6 \times 10^{21}$

Mass of Ra required = $2.6 \times 10^{21} / 6.02 \times 10^{23} \times 226$ ✓
= 0.98g = 9.8×10^{-4} *kg* ✓

Tip: To convert a number of atoms to a mass:
- ***Divide the number of atoms by Avagadro's number***
- ***Multiply the answer by the atomic mass***

This gives the mass in grammes.

Example 2: Uranium-235 undergoes fission according to the equation

$$^{235}_{92}U + ^{1}_{0}n \rightarrow ^{96}_{37}Rb + ^{138}_{55}Cs + 2^{1}_{0}n$$

(a) Calculate the energy given out by this reaction, giving your answer in electronvolts

(b) Calculate the energy given out by the fission of 1kg of uranium-235, giving your answer in MJ.

Masses: $^{235}_{92}U$ ***: 235.044u*** $^{96}_{37}Rb$ ***: 95.933u***

$^{138}_{55}Cs$ ***: 137.920u*** $^{1}_{0}n$ ***: 1.009u***

$c = 3.0 \times 10^{8}$ ***ms***$^{-1}$

(a) Mass defect $= (235.044 + 1.009 - 95.933 - 137.920 - 2 \times 1.009)u = 0.182u$

Mass defect in kg $= 0.182 \times 1.66 \times 10^{-27} = 3.0212 \times 10^{-28}$ *kg*

Energy given out in joules $= 3.0212 \times 10^{-28} \times (3 \times 10^{8})^{2} = 2.719 \times 10^{-11} J$

Energy given out in electronvolts $= 2.719 \times 10^{-11} / 1.602 \times 10^{-19} = 1.7 \times 10^{8}$ *eV*

> ***Tip: When you convert from J to eV, you should always get a larger number. If you do not, you have probably multiplied by*** 1.602×10^{-19} ***instead of dividing.***

(b) Need to find number of atoms of uranium-235 in 1kg:

> ***Tip: To convert a mass to a number of atoms:***
> - ***divide the mass in grammes by the atomic mass***
> - ***multiply the answer by Avagadro's number***

Number of atoms $= 1000/235 \times 6.02 \times 10^{23} = 2.56 \times 10^{24}$

So total energy released = no. of atoms × energy released per atom

$= 2.56 \times 10^{24} \times 2.719 \times 10^{-11} J$

$= 6.96 \times 10^{15} J$

$= 6.96 \times 10^{9} MJ$

Typical Exam Question

In a nuclear fusion reaction, a nucleus of deuterium (hydrogen – 2) coalesces with a tritium nucleus (hydrogen – 3). A helium nucleus is formed, together with another particle.

(a) Identify the other particle formed and write an equation for the reaction. [2]

(b) Calculate the energy evolved in the reaction, giving your answer in MeV [5]

Combined mass of deuterium and tritium nuclei = 5.031u

Combined mass of the particles produced in the reaction = 5.011u

1u = 1.661 × 10^{-27}**kg**

Speed of light $c = 3.000 \times 10^{8}$ **ms**$^{-1}$

Electronic charge $e = 1.602 \times 10^{-19}$**C**

(a) $^{2}_{1}H + ^{3}_{1}H \rightarrow ^{4}_{2}He + ?$

Since the Z and A values must balance, the particle produced has a nucleon number of 1 and proton number of 0 ⇒ it is a neutron. ✓

So the equation is $^{2}_{1}H + ^{3}_{1}H \rightarrow ^{4}_{2}He + ^{1}_{0}n$ ✓

(b) Mass defect $= 5.031 - 5.011 = 0.020u$ ✓

Mass defect in kg $= 0.020 \times 1.661 \times 10^{-27} = 3.322 \times 10^{-29}$ *kg* ✓

Energy produced $= 3.322 \times 10^{-29} \times c^{2} = 2.9898 \times 10^{-12} J$ ✓

Energy produced in eV $= 2.9898 \times 10^{-12} / 1.602 \times 10^{-19}$ ✓

$= 1.87 \times 10^{7} eV$

$= 18.7$ *MeV* ✓

Fission and fusion

Nuclear fission and fusion both release a great deal of energy (see previous examples); their use as power supplies is therefore of great interest. Both fission and fusion produce a much greater amount of energy per kilogram of fuel than do "conventional" energy sources such as fossil fuel combustion; for example, the fission of 1 kilogram of uranium-235 releases the same amount of energy as the combustion of about 25000 kg of coal.

Fission

Fission of **uranium – 235** is used to produce power. In this reaction, a neutron hits the U-235 nucleus, which then splits to produce two smaller nuclei, more neutrons and energy (mainly in the form of kinetic energy of the daughter nuclei and neutrons). The smaller nuclei formed may vary; one example is shown in example 2 on the left and another is shown below:

$$^{235}_{92}U + ^{1}_{0}n \rightarrow ^{144}_{56}Ba + ^{90}_{36}Kr + 2^{1}_{0}n$$

The number of neutrons may also vary; it is generally two or three, with an average of about 2.6.

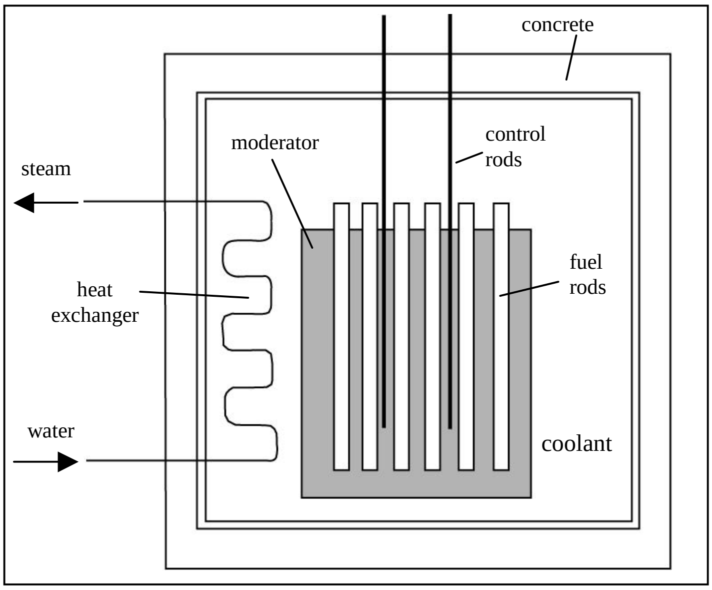
The neutrons given off in the fission of the U-235 atom may then go on to hit other atoms, causing them to undergo fission in turn, which produces more neutrons, causing more fission etc. This is known as a **chain reaction**. To produce power, the chain reaction must be maintained, but controlled. To maintain the chain reaction, a minimum of one neutron from each fission reaction should go on to cause another fission; to control it, there should not be more than one neutron causing further fission. An uncontrolled chain reaction produces a nuclear bomb.

To maintain a controlled chain reaction, the following problems must be overcome:

- If the amount of fission material is too small, too many neutrons will escape. The minimum acceptable amount of fission material is called the **critical size**.
- The neutrons required to cause fission are slow neutrons, but the neutrons produced by fission are quite fast. So the neutrons produced have to be slowed down.
- Naturally occurring uranium contains only a small proportion of U-235; the commonest isotope, U-238, absorbs neutrons without undergoing fission. So the proportion of U-235 needs to be increased before uranium can be used as fuel.

Fission is carried out in a **thermal reactor** (fig 2). The components of this reactor are used to ensure a controlled chain reaction is maintained.

Fig 2. Thermal Reactor



- The **fuel rods** are made of enriched uranium – that is, natural uranium with extra U-235 added.
- The **moderator** slows down the neutrons produced in fission so that they can stimulate further fission. It is made of graphite (or sometimes water). The neutrons are slowed down by their collisions with carbon nuclei in the moderator; some of their energy is transferred to the moderator, which gets hot. The slow neutrons are known as "thermal neutrons", because they move at speeds associated with thermal motion; they give the thermal reactor its name.
- The **control rods** are made of a substance, such as boron, that absorbs neutrons. They are used to control the chain reaction, to ensure too many neutrons do not cause fission. They can be raised or lowered to speed up or slow down the reaction.
- The **coolant** - usually water or pressurized carbon dioxide - is used to remove energy from the system. The heat generated in the fuel rods is transferred to the coolant by conduction; this transmits it to the heat exchanger, where it is used to convert water to high pressure steam, which is used to drive a turbine to produce electricity.
- The **concrete shielding** absorbs the nuclear radiation; it is necessary because of the danger nuclear radiation poses to living creatures.

There are **environmental hazards** associated with nuclear reactors; they include:

- If radioactive gases or dust escape into the atmosphere, they can be readily absorbed by humans and animals via food, water or breathing.
- Used, "spent" fuel rods contain many radioactive decay products; accordingly they must be stored securely to prevent their being a hazard. Some of these decay products have very long half-lives, and hence must be stored for thousands of years.
- If the chain reaction in a fission reactor is not controlled, the reactor may act as a bomb.

The **benefits** of fission reactors include:

- A great deal of energy is produced from a small amount of fuel
- Fission does not produce the gaseous waste products (carbon dioxide, nitrogen and sulphur oxides) associated with combustion of fossil fuel.
- Suitable fuel is not in such short supply as fossil fuels.

Fusion

Fusion involves two light nuclei combining to form a heavier one, together with energy and one or more neutrons. It is the process which produces energy in the sun. One such reaction involves the fusion of deuterium (hydrogen-2) and tritium (hydrogen-3) nuclei to form a helium nucleus, a neutron and energy:

$$^2_1H + ^3_1H \rightarrow ^4_2He + ^1_0n$$

Fusion reactors are not yet in existence; it is much more difficult to achieve and control fusion than fission. This is because it is necessary to overcome the repulsion between the nuclei. This requires a very high temperature – 100 million K or above. Normal containers cannot hold anything as hot as this; containing this material by magnetic fields is one possibility.

Fusion has a number of advantages:

- Fusion is a more productive energy source than fission per kilogram of material.
- The raw materials for fusion can be obtained from sea water.
- The waste products are not radioactive.
- Uncontrolled chain reactions cannot develop.

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Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

(a) Explain the purpose of the moderator in a thermal reactor. [2]

To slow down neutrons ✓ to stop the reaction going too fast ✗ 1/2

This is a common mistake – the neutrons have to be slowed down to allow the chain reaction to take place, not to control it.

(b) Suggest a suitable radioactive source for this reactor [1]

Uranium ✗ 0/1

Not specific enough – only U-235 is suitable, not U-238

(c) Suggest two advantages and two disadvantages of the use of thermal nuclear reactors rather than fossil fuels to generate electricity. [4]

Disadvantages: dangerous ✗ waste must be stored for a long time ✓
Advantages: no pollution ✗ doesn't rely on fossil fuels ✓ 2/4

"Dangerous" is not specific enough – how and why is it dangerous? "No pollution" is not specific enough either – the candidate should have specified which forms of pollution are reduced.

(d) Explain how the energy released by a nuclear fission reaction is calculated, including a suitable equation. How reliable is this calculation? [4]

Using $E = mc^2$ ✓
There may be rounding errors, making it inaccurate ✗ 1/4

The candidate should have realised this was not enough for 4 marks! Questions on the reliability of a calculation must always be answered specifically to the context – rounding error can apply to any calculation in physics.

Examiner's Answers

(a) Slows neutrons ✓ so that they are more likely to cause fission. ✓
(b) Uranium-235 ✓
(c) Advantages: high energy density, does not produce pollutants carbon dioxide/nitrogen & sulphur oxides. ✓✓
Disadvantages: Any two of: waste products dangerous, danger of meltdown, storage of waste products a problem ✓✓
(d) There is a mass defect for the reaction ✓ which allows the energy released to be calculated ✓ by using $E = mc^2$ ✓. This calculation ignores the energy produced by later reactions of fission products and so may or may not be reliable. ✓

Questions

1. Explain what is meant by binding energy and mass defect, and state an equation connecting them.
2. Sketch the curve of binding energy per nucleon against proton number, and explain how it can be used to predict which atoms will undergo fission and which will undergo fusion.
3. Explain the principles of operation of a thermal reactor.
4. Explain what is meant by nuclear fusion, and give **two** advantages of fusion over fission as an energy source.
5. $^{238}_{92}U$ decays by alpha-emission to give thorium. Calculate the energy emitted, giving your answer in MeV.
(Masses: U-238, 238.0508u; Th-234, 234.0437u; He-4, 4.0026u)
$1u = 1.661 \times 10^{-27}$ kg; $c = 3.0 \times 10^8$ ms^{-1}

Answers

1 – 4 can be found in the text.

5. Mass defect = (238.0508 – 234.0437 – 4.0026_u = 0.0045u
Mass defect = $0.0045 \times 1.66 \times 10^{-27} = 7.47 \times 10^{-30}$ kg
Energy in joules = 7.47×10^{-30} kg $\times c^2 = 6.723 \times 10^{-13}$
Energy in eV=$6.723 \times 10^{-13} / 1.602 \times 10^{-19} = 4.20 \times 10^6$eV = 4.20 MeV

Physics Factsheet

January 2001 Number 11

Radioactivity – an introduction

This Factsheet will explain the nature, properties and effects of radioactive emissions, the concept of half-life and the hazards and benefits of radioactivity.

What is radioactivity?

Radioactivity involves the spontaneous (in other words, occurring without outside interference) emission of an alpha or beta particle from the nucleus of an atom, causing the proton number (see box) of the atom to change. An individual radioactive decay therefore always involves one element changing to another element.

Radioactive decay may also be accompanied by gamma emission. This is how the nucleus rids itself of excess energy if it is in an excited state after emitting an alpha or beta particle.

The properties of alpha and beta particles and gamma rays are shown in Table 1. As the table shows, they differ significantly in their
- range – how far they travel, or how easily they are stopped
- ionising ability – their ability to "knock" electrons from atoms that they collide with, turning them into ions.

These are important in considering health risks

Table 1. Properties of alpha, beta and gamma radiation.

Type of radiation	Consists of	Range	Ionising Properties
alpha (α)	helium nuclei – i.e. 2 protons + 2 neutrons	5 cm in air – can be stopped by a thick sheet of paper	Highly ionising
beta (β^-)	fast electrons	Up to a few metres in air – can be stopped by a few millimetres of aluminium	Less than α
gamma (γ)	very high frequency (and so high energy) electromagnetic waves	Significantly reduced by several metres of concrete, or several cm of lead	Weak – much less than β

***Tip**: In beta emission, the electron comes **from the nucleus**. It is **not** one of the electrons from the atom.*

What sort of atoms decay?
Atoms that decay radioactively are known as **unstable**; those that do not are **stable**. Atoms may be unstable because:
- they are too large (with a larger nucleus than lead)
- the balance between protons and neutrons is not right. In small stable atoms, there are roughly the same number of protons and neutrons, whilst in large stable atoms, neutrons always outnumber protons.

This will be covered in more detail in later work on radioactivity.

Atoms – a reminder
An atom contains **protons**, **neutrons**, and **electrons**; the properties of these are shown in the table below.

Particle (symbol)	Mass (atomic mass units)	Charge (units of 1.6×10^{-19}C
proton (p)	1	+1
neutron (n)	1	0
electron (e)	≈ 0	-1

The **nucleus** of the atom contains the protons and neutrons, which are known collectively as **nucleons**. **The nucleus is the only part of the atom involved in radioactivity**.

Two numbers are used to describe the particles in the nucleus:
- **A** – the **nucleon number** (or **mass number**)
- **Z** – the **proton number** (or **atomic number**)

So, if an atom has A = 13 and Z = 6, then it would have 6 protons, and 7 neutrons (since protons + neutrons = A). **The proton number tells you which element the atom is** – so for example, any atom with proton number 6 is a carbon atom.

We write an atom of element X with nucleon number A and proton number Z as ${}^{A}_{Z}X$ - so an atom of oxygen with 8 protons and 8 neutrons in its nucleus is written ${}^{16}_{8}O$.
- **Isotopes** of an element have the same proton number, but different nucleon numbers – so they have the same number of protons but different numbers of neutrons. Common examples of isotopes are ${}^{14}_{6}C$ and ${}^{12}_{6}C$, and ${}^{35}_{17}Cl$ and ${}^{37}_{17}Cl$.

Where does radioactivity come from?

Radioactivity comes from both natural and man-made sources. Man-made, "artificial" elements with very large nuclei are always radioactive, but many naturally occurring elements (or particular isotopes of elements) are radioactive too.

This naturally occurring radiation means that we are exposed to a low level of radiation constantly; this is called **background radiation**.

Sources of background radiation include:
- *rocks, such as granite*
- *radon gas, which is formed in the ground*
- *cosmic rays, which come from space*
- *artificially produced radioisotopes*

Effect of radioactive decay on the nucleus – nuclear equations.

- Alpha decay removes 2 protons and 2 neutrons from the nucleus – so Z, the proton number, decreases by 2, and A, the nucleon number, decreases by 4.
- In beta decay, a neutron in the nucleus emits an electron and changes to a proton. So Z increases by 1 and A stays the same.
- Gamma radiation only involves the emission of energy, and so does not change either A or Z.

Radioactive decay can be represented in **nuclear equations**:

eg: α decay of radon 204:

$$^{204}_{86}\text{Rn} \rightarrow {}^{200}_{84}\text{Po} + {}^{4}_{2}\alpha$$

The atom we start with – radon 204	The atom produced after decay – polonium 200	alpha particle

β decay of beryllium 10:

$$^{10}_{4}\text{Be} \rightarrow {}^{10}_{5}\text{B} + {}^{0}_{-1}\beta$$

The α particle may also be written as $^{4}_{2}\text{He}$ (as it is a helium nucleus).

The β particle may also be written as $^{0}_{-1}\text{e}$ (as it is an electron)

Tip*: Check your finished equation is right by checking that the A values add up to the same on both sides, and the Z values add up to the same on both sides.*
Always put the A and Z values on for the emitted particles as well as the atoms. This helps you to make sure you get the equation right.

Radioactive decay series
In some cases, the atom produced is also radioactive, and so decays in its turn. This process continues until an atom is produced that will not decay. If the initial atom has a large nucleus, then the final, non-radioactive atom in the series will be a stable isotope of lead.

eg: $^{238}_{92}\text{U} \xrightarrow{\alpha} {}^{234}_{90}\text{Th} \xrightarrow{\beta} {}^{234}_{91}\text{Pa} \xrightarrow{\beta} {}^{234}_{92}\text{U} \xrightarrow{\alpha} {}^{230}_{90}\text{Th} \xrightarrow{\alpha} {}^{226}_{88}\text{Ra}...$

Artificial radioactivity
Radioactive isotopes can be created by bombarding naturally occurring atoms with particles such as neutrons (used in a nuclear reactor), protons or α particles. Nuclear equations can be written for this type of process in exactly the same way:

eg: $^{63}_{29}\text{Cu} + {}^{1}_{0}\text{n} \rightarrow {}^{64}_{29}\text{Cu} \rightarrow {}^{64}_{30}\text{Zn} + {}^{0}_{-1}\beta$

Some artificially produced isotopes decay by emitting a **positron** – which is a particle identical to an electron, except that it is positively charged (in fact, it is the **antiparticle** of the electron). This is known as positron emission, or β^+ decay. In positron emission, a proton emits a positron, thus changing into a neutron, so the proton number decreases by 1, and the nucleon number remains unchanged.

eg: Fluorine is bombarded with α particles to produce sodium –22:

$$^{19}_{9}\text{F} + {}^{4}_{2}\alpha \rightarrow {}^{22}_{11}\text{Na}$$

Sodium – 22 then decays by positron emission to form Neon – 22

$$^{22}_{11}\text{Na} \rightarrow {}^{22}_{10}\text{Ne} + {}^{0}_{1}\text{e}$$

Typical exam question
The following nuclear equation represents the decay of uranium to thorium: $^{238}_{92}\text{U} \longrightarrow {}^{234}_{90}\text{Th} + {}^{y}_{z}\text{X}$

(a) Determine the values of y and z [2]
(b) Identify the particle X [1]
(c) Q represents the energy released in this reaction. Which two forms does this energy take? [2]
(d) The thorium also decays by β^- emission forming an isotope of palladium (Pa). Write the nuclear equation for this decay. [2]

(a) y = 238 – 234 = 4 ✓ z = 92 – 90 = 2 ✓
(b) The particle has 4 nucleons and 2 protons
It is an alpha particle or He nucleus. ✓
(c) Kinetic energy of the alpha particle ✓Gamma radiation ✓
(d) $^{234}_{90}\text{Th} \rightarrow {}^{234}_{91}\text{Pa}$ ✓ + β ✓ + *(energy (γ))*

Decay law and half life
Radioactive decay is a **random** process – it is not possible to predict when a particular nucleus will decay. We can get a picture of how this works using another "random" process – throwing dice.

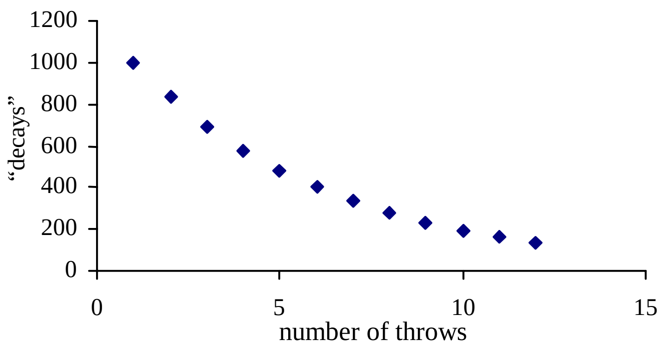
Imagine you have a large number (say 6000) of normal dice. Each dice represents a radioactive atom, and we will say a dice has "decayed" (to give a stable product) if it shows a 1 when it is thrown. Of course, the dice are not a perfect model – radioactive decay goes on continuously, not just at set time intervals, and may produce radioactive products. However, it is close enough to be useful.

All the dice are thrown. About one sixth of the dice, or about 1000 dice, will show a 1 – so these atoms have decayed.

There are 5000 "undecayed" dice left. They are thrown again – again, about one sixth of them will show a 1, and decay. The decayed dice are removed, and the process is repeated.

The numbers of dice we'd expect to decay on each throw are shown in the table and graph below:

Throw	No. of Decays	Throw	No. of Decays
1st	1000	7th	335
2nd	833	8th	279
3rd	694	9th	233
4th	579	10th	194
5th	482	11th	162
6th	402	12th	135



Key points to note:

- We can predict that the actual results of throwing the dice would give results very much like these. However, we have no idea which dice will "decay" on any one throw, or how long a particular dice will take to "decay".
- The probability of any dice "decaying" is always the same – it is $\frac{1}{6}$. So the number of dice decaying each time will be about $\frac{1}{6}$ of the dice that are left.
- The graph produced by the dice has a characteristic shape – it is called an exponential decay curve. (see Factsheet 10 – Exponentials and Logarithms – for more on exponentials)

Radioactivity behaves very much like the dice – there is a constant probability of a nucleus decaying, so the number of nuclei decaying is proportional to the total number of them. The number of decays per second of a sample is called its **activity**, and is measured in bequerels (Bq)

The number of atoms decaying over a given period is a constant fraction of (or is proportional to) the number remaining.

This is expressed in an equation as **A = λN**
where A = activity, λ = decay constant (which is the probability of any given atom of that element decaying in one second) and N = no. of undecayed atoms.

This equation leads to an **exponential decay curve**.

Tip: You must remember that the atoms decaying do not disappear – they change into another element, which may or may not be radioactive.

An exponential decay curve has some very important properties; these will also come up elsewhere in Physics.

- Activity never decreases to zero
- There is a **constant half life** – for a given radioactive element, the time taken for the activity to halve will always be the same. (So, for example, it would take the same time for activity to decrease from 400 Bq to 200 Bq, as for activity to decrease from 200Bq to 100Bq
- The **half-life** in seconds is given by $T_{½} = \frac{0.69}{\lambda}$, where λ is the decay constant. (This equation will be justified in later studies)

Calculations involving half life

Calculations involving half life may require you to:

- determine half-life from a graph
- determine half life from count rates
- determine count-rates or time, given the half life.

The following examples illustrate these.

Example 1. The table below shows the count rate for a radioactive isotope. The background count rate is 0.6 counts per second.

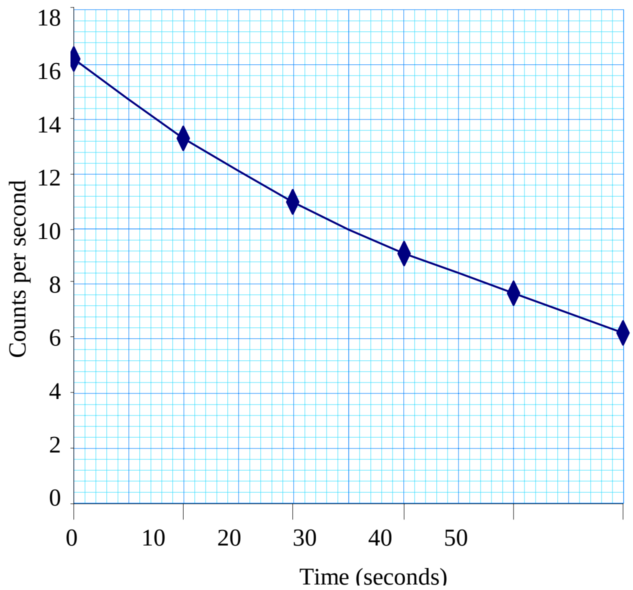
Time (s)	***0***	***10***	***20***	***30***	***40***	***50***
Count rate (counts per second)	***16.7***	***13.8***	***11.6***	***9.7***	***8.2***	***6.8***

Plot a graph of corrected count rate against time and use it to determine the half life of the isotope.

*Tip: If you are told to plot a graph of something against something, the thing given **first** (in this case, count rate) goes on the **y-axis**.*

The table below shows the corrected count rates:

Time (s)	*0*	*10*	*20*	*30*	*40*	*50*
Count rate (counts per second)	*16.1*	*13.2*	*11.0*	*9.1*	*7.6*	*6.2*



Now we measure two or more half-lives from the graph. For example, we could choose to find the time taken for the count rate to decline from 16 to 8, and from 14 to 7. We then average the values.

From graph:
time taken to decline from16 to 8 is 37 seconds.
time taken to decline from 14 to 7 is 45 – 7.5 = 37.5 seconds.

So our estimate is the average of these values – 37.25=37 sec(2 SF) (since the original data was given to 2 SF, it would not be appropriate to use any greater accuracy in the answer).

*Tip: **You must** always **use** corrected count rates **in any half-life calculations. If you are told the background count, use it!***

Example 2. A sample of carbon 14 has an activity of 40 Bq
a) After 17 100 years, the activity of this sample will have fallen to 5Bq. Calculate the half life of carbon 14.

b) After how many more years will the activity of this sample have fallen to approximately 0.078Bq?

a) We need to work out how many half-lives are required for the activity to fall to 5Bq.

$$40 \xrightarrow{T_{1/2}} 20 \xrightarrow{T_{1/2}} 10 \xrightarrow{T_{1/2}} 5$$

So 17100 years = 3 half-lives.
So half-life = 17 100 ÷3 = 5700 years.

b) 5 → 2.5 → 1.25 → 0.625 → 0.3125 → 0.15625 → 0.078125
So it takes another 6 half-lives = 34 200 years.

Typical Exam Question
The half life of a sample of radioactive material is related to its decay constant by the equation:

$$T_{1/2} = \frac{0.69}{\lambda}$$

(a) Explain the meaning of the symbols:
(i) $T_{1/2}$ [1]
(ii) λ [1]
(b) A sample of ^{24}Na has a half-life of 234 hours.
Calculate the radioactive decay constant for ^{24}Na [2]

(a) (i) $T_{1/2}$: Half-life, the average time for the number of undecayed atoms or nuclei to be reduced to half their initial number. ✓
(ii) λ: Decay constant, the constant of proportionality in the relationship: activity $\propto$ number of undecayed atoms ✓
(b) $\lambda = \ln 2 /(234 \times 3600)$ ✓ $= 8.2 \times 10^{-7} s^{-1}$ ✓

Typical Exam Question
(a) Sketch a diagram of the apparatus that could be used in a school laboratory to determine the half-life of a radioactive sample. [3]
(b) State the measurements that are required to accurately determine the half-life. [2]
(c) Explain how the measurements would be used to determine the half-life for the sample. [3]
(a) Diagram should include radioactive source✓ and GM tube✓ connected to ratemeter✓
(b) For this sample the count rate should be recorded at intervals of 10s (5-20s) ✓ for 300s. ✓
(c) Plot a graph of count rate against time✓
Find, from the graph, the time for the count rate to half ✓
Repeat using different intervals of halving (1600 to 800, 1000 to 500 etc.) ✓ Average the results.

Experiments with radioactivity

A **Geiger-Müller tube** (G-M tube) attached to a **ratemeter** is used to detect radioactivity in the school laboratory. It is used to measure the radioactive count-rate per second (which is proportional to the number of emissions per second).

Some errors arise in the use of the GM tube because after it has registered one count, there is a short interval (known as the **dead time**) before it can register another, so any decays occurring in this period are not registered. This problem is most noticeable when the count rate is high.

Radioactive sources for experiments are always supplied in a holder, and have low levels of activity. When not in use, they are stored inside a lead "castle", in a wooden box; accordingly they will prevent no danger when stored. The following **safety precautions** must always be observed when handling radioactive sources:

- always lift with forceps
- hold so that the open end is directed away from the body
- never bring close to the eyes

In all experiments, it is necessary to first measure the **background radiation** for a short time and obtain an average value. This value is then subtracted from all future readings to give the count rate due to the source alone – the **corrected count rate**.

In any experiment aiming to measure the count rate for just one type of radiation (in other words, for just α, just β or just γ), it is necessary to ensure that the other types of radiation do not "interfere":

- To obtain γ alone, a cover disc of aluminium over the source will absorb any α or β produced (NB: γ is never produced alone, so this will always be necessary).

- An α or β source can be selected that does not emit γ radiation, but it is also necessary to consider any other products in the radioactive decay series; if an α emitter produces a decay product that is also radioactive, the count-rate of the decay product will also be included in any measurements. This is addressed by selecting a source for which all the decay products in the series have much longer half-lives (and so a low, roughly constant activity rate) over the time of the experiment, or much shorter half lives (so almost all of the atoms decay within a very short time-scale).

Exam Hint: *Examination questions about experiments require essential practical details, such as precautions, how frequently measurements are taken and sources of error.*

Absorption of α radiation.
After the background radiation has been measured, the source is intially placed close to the GM tube and the average reading noted over a short period. This is repeated, moving the GM tube away from the source in 5mm steps, until the count rate has returned to the background level (or the corrected count rate is zero. Materials such as paper or metal can be placed between the source and the counter to demonstrate absorption; in this case the GM tube must be less than 5cm from the source.

Absorption of β radiation
The same procedure as for α radiation can be followed, except that the steps by which the GM tube is moved away need to be longer, since the range of β radiation is much greater. Various thicknesses of aluminium foil can be placed between the source and the counter to determine the required thickness for total absorption.

Absorption of γ radiation
Gamma radiation is only absorbed to any extent by lead; the thickness of lead required can be determined in the same fashion as the previously described experiments.

Range of γ radiation in air
Gamma rays are not absorbed to any great extent by air; their intensity falls off with distance according to an inverse square law – in other words, the intensity is inversely proportional to the square of the distance from the source. This gives the equation:

$$I = \frac{k}{d^2}, \text{ where k is a constant.}$$

This is demonstrated by placing a GM tube at various distances from a gamma source and measuring the count rates. The count rates are corrected for background radiation. A graph is then plotted of corrected count rate (y-axis) against $\frac{1}{d^2}$. This should give an approximately straight line.

Errors – resulting in the graph not being exactly a straight line – are due to the source itself having a size – in other words, not being a "point source" to the GM tube's dead time., and to the random nature of decay.

Measuring half life
After the background count has been measured, the source is placed close to the GM tube. Counts are taken at 10 second intervals, and a graph of corrected count rate (y-axis) against time (x-axis) can be produced. From the graph, a number of values for the half-life can be found; an average of these produces a suitable estimate. (In later work, an alternative, more accurate approach will be used).

Uses of radioactivity

Carbon dating

Carbon-14 is a naturally occurring beta-emitter with a half-life of 5700 years. It is formed in the atmosphere from nitrogen, due to the action of cosmic rays, and becomes incorporated in radioactive carbon dioxide.

During photosynthesis, plants and trees take in carbon dioxide from the atmosphere; this includes carbon-14. The amount of carbon-14 present as a proportion of the total amount of carbon will be, on average, the same in a living plant as in the atmosphere as a whole.

However, when the plant dies, it stops interacting with the atmosphere, so it doesn't acquire any more carbon-14. The activitity level then declines exponentially. By measuring the residual activity, it is possible to estimate how many half-lives there have been since the plant died, and hence how long ago it lived.

Carbon-14 is particularly suitable for this use due to:
- its presence in all living things
- the length of its half life – sufficiently short that changes can be observed over thousands of years, but sufficiently long that there is still significant residual activity after this period.

This method assumes that the proportion of carbon-14 in the atmosphere has stayed the same; this depends on whether the amount of cosmic rays penetrating the atmosphere was the same.

Radioactive tracers

Radioactive tracers are used to follow the path of a compound in a system such as pipelines or the human body. They rely on the fact that radioactive isotopes behave identically to non-radioactive ones in physical and chemical processes. For example, a radioactive tracer can be used to detect a leak in a pipe, since the count-rate will increase where the leak occurs as the pipe will block α and β emissions.

A γ emitter would not be suitable, since the pipe would not block this. Isotopes used for this purpose need to have a suitable half-life, so that the count rate will not become so low as to be almost undetectable during the course of the investigation.

Sterilization

Gamma rays can be used to sterilize medical instruments or keep food fresh for a longer period.

Radiotherapy -cancer treatment

Radiotherapy involves using gamma sources to attack cancer cells. It relies on the cancerous cells being more affected by the radiation than the normal ones, but obviously the normal cells are affected too, so it does produce some unpleasant side effects, like those described in the "Dangers of radioactivity" box. Again, a short half-life is required.

Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

Top number → $^{238}_{92}U$ ← Bottom number **and $^{235}_{92}U$ are radioactive isotopes of Uranium.**

(a) State the correct terms, and meanings of, the 'top number' and 'bottom number'. [4]

top number: number of protons + neutrons ✓
bottom number: number of protons ✓ 2/4

> Although the student knows what the numbers mean, s/he has lost out by not giving the correct terms – not reading carefully, perhaps?

(b) Explain the meaning of the term 'isotope'. [2]

atoms with different numbers of neutrons ✓ 1/2

> To obtain both marks, the candidate needs to make it clear that the isotopes have the same number of protons. This could be written down directly, or the candidate could make reference to A and Z.

(c) U-238 decays via α-emission to produce an atom of thorium(Th). Write a nuclear equation to represent this decay. [3]

$^{238}_{92}U \rightarrow ^{234}_{90}Th + \alpha$ ✓✓ 2/3

> The two marks awarded were for calculation of A and Z for thorium. The candidate needed to indicate the values of Z and A for the α particle to obtain full marks; clearly s/he knew them, since the other calculation was correct.

(d) Uranium-238 has a half-life of 4.5×10^9 years. Calculate the time required for the activity of a sample of Uranium-238 to decrease from 6.04×10^4 Bq to 1.8875×10^3 Bq.

$6.04 \times 10^4 \div 1.8875 \times 10^3 = 32$ ✓
$2 \times 2 \times 2 \times 2 \times 2 = 32$, so 5 half-lives ✓ 2/3

> What the candidate has done is correct, and clearly presented. But s/he has missed out on the final mark by not reading the question carefully – it asks for the time in years, not the number of half-lives. The candidate should be surprised not to use all the information.

Examiner's Answers

(a) *Top: nucleon / mass number.* ✓
It's the no. of nucleons or protons + neutrons in nucleus ✓
Bottom: atomic number / proton number ✓
It's the number of protons in nucleus ✓

(b) *Atoms with the same number of protons* ✓
but different numbers of neutrons in their nuclei. ✓

c) $^{238}_{92}U \rightarrow ^{234}_{90}Th + ^{4}_{2}\alpha$ *(1 each for 234, 90 and 1 for 4 and 2)*

d) $6.04 \times 10^4 = 1.8875 \times 10^3 \times 2 \times 2 \times 2 \times 2 \times 2$ ✓*(method)*
so 5 half-lives ✓. *So* $5 \times 4.5 \times 10^9 = 2.25 \times 10^{10}$ *years* ✓

Dangers of radioactivity

Since α radiation is so easily absorbed, it is not dangerous unless the radioactive source is inside the body. Although β radiation is more penetrating, most of its energy is usually absorbed by clothes, and it is easy to protect people further by using aluminium shielding. The greatest danger arises from gamma radiation; although it is not strongly ionising, it can penetrate deeply into the body.

Damage from radiation can include:
- radiation burns (like normal burns, but caused by gamma rays)
- hair loss
- radiation sickness
- damage to reproductive organs
- delayed effects such as cancer and leukemia

The level of danger depends on the amount of radiation absorbed; people likely to be exposed to radioactive materials, such as workers in nuclear power plants, have their radiation dosage carefully monitored to ensure it does not exceed safe levels.

The hazard represented by a particular radioisotope is therefore dependent on:

- the nature of its emissions, and of the emissions of its decay products
- its level of activity
- its half-life, since long half-life radioisotopes will continue to be highly active for a long period of time, and hence potentially be a danger for this time.

This has implications for the disposal of nuclear waste, which includes long-half-life isotopes. The canisters used to contain nuclear waste need to be resistant to naturally occurring phenomena like landslides or earthquakes, since the contents would still represent a danger for many years to come.

Qualitative Questions

1. Give two sources of background radiation
2. Give two uses for radioactive elements, other than carbon dating.
3. Explain how carbon dating works.
4. Describe the penetrating power of α, β and γ radiation.
5. Explain what is meant by "half-life"
6. Explain what is meant by "activity", and give its units
7. Which form of radiation is potentially the most dangerous to human beings?
8. Explain why γ emission occurs

The answers to these questions may be found in the text.

Quantitative Questions

1. Write nuclear equations for the following decays:

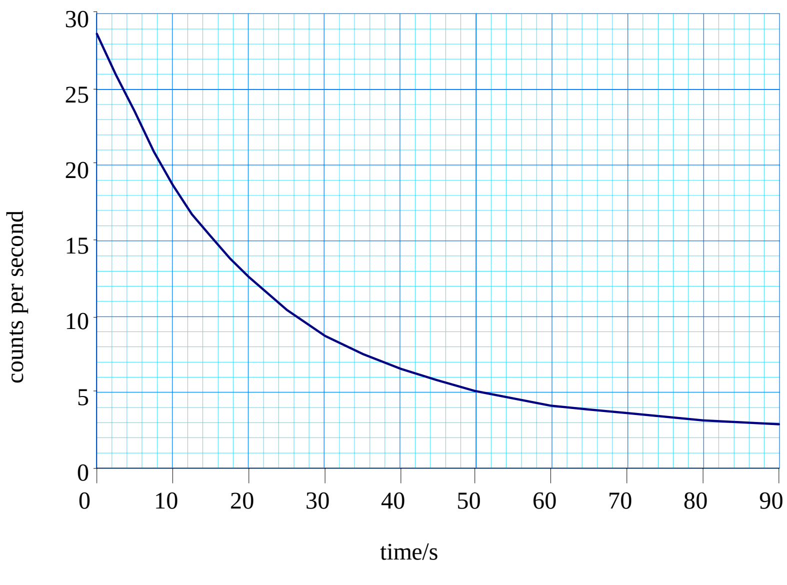
a) β emission from a $^{60}_{27}Co$ nucleus to produce Nickel (Ni)

b) α emission from a $^{214}_{84}Po$ nucleus to produce Lead (Pb)

c) $^{63}_{29}Cu$ absorbs a neutron, then decays by β emission to form zinc (Zn)

2. An isotope has a half-life of 12 hours. Calculate its decay constant.

3. The graph below shows the decline of activity with time for a radioisotope. Calculate its half-life.
The background count is 2.6.



Hint: the background count must be allowed for – so, for example, find the time required for the count to decline from 24 + background to 12 + background.

4. A radioisotope has a half-life of 12 hours.
Its initial activity is 1.6×10^{12} Bq.
Find the time required for its activity to decline to 3.125×10^{9} Bq.

5. The activity of a radioisotope declines from 920Bq to 7.1875Bq in 1 hour 10 minutes. Find its half life, giving your answer in minutes.

Answers

1. a) $^{60}_{27}Co \rightarrow ^{60}_{28}Ni + ^{0}_{-1}\beta$

b) $^{214}_{84}Po \rightarrow ^{210}_{82}Pb + ^{4}_{2}\alpha$

c)) $^{63}_{29}Cu + ^{1}_{0}n \rightarrow ^{64}_{30}Zn + ^{0}_{-1}\beta$

2. 1.60×10^{-5} (3SF)

3. 14 – 15 seconds.

4. 108 hours (or 4 days 12 hours)

5. 10 minutes

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Physics Factsheet

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Radioactivity II

This Factsheet will discuss the quantitative treatment of radioactive decay and explain how to predict which nuclei will decay and how they will decay.

Before studying this Factsheet, you should be familiar with basic concepts in radioactivity (covered in Factsheet 11); these include:
- **the nature and properties of α, β, and γ radiation**
- **background radiation and how to correct for it**
- **half-life**

It would also be helpful to be acquainted with exponentials and logarithms (Factsheet 10).

Decay law – a review
From earlier studies, you should recall that:
- radioactive decay is a random process;
- the probability of a nucleus decaying is constant.

These facts lead to:
- The number of nuclei decaying per second (the **activity)** is proportional to the total number of nuclei in the sample.
  (This is like saying that you'd expect the total number of sixes obtained by rolling a lot of dice would be proportional to the number of dice – you'd expect twice as many sixes with 200 dice as with 100 dice)

This can be expressed as an equation:

$A = \lambda N,$

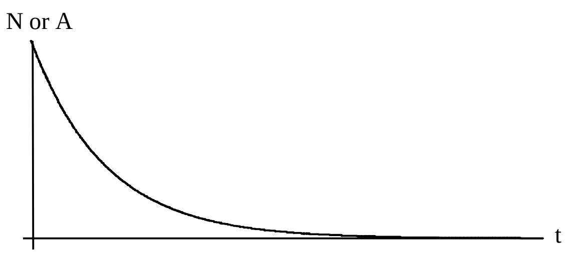
A = activity
λ = decay constant (= probability of 1 atom decaying in 1 second)
N = number of undecayed atoms.

Exponential decay
The decay equation tells us that the number of nuclei decaying is proportional to the total number of nuclei; this may also be written as

rate of decrease of N = λN

(or for those studying A2 Maths: $\frac{dN}{dt} = -\lambda N$)

This type of equation (which will also be encountered in other areas of Physics, notably charge decay for a capacitor) leads to an **exponential decay curve** when N (number of undecayed atoms) or A (activity) is plotted against time:



Exponential decay curves have some important properties:
- For small values of t the curve falls off quickly but this slows down as t becomes big; the gradient never = 0 and the curve never cuts the t axis;
- There is a **constant half life** – for a given radioactive element, the time taken for the activity to halve will always be the same. (So, for example, it would take the same time for activity to decrease from 400 Bq to 200 Bq, as for activity to decrease from 200Bq to 100Bq

Equation for exponential decay curves
Equations for exponential curve involve the number "e". This is a number between 2 and 3; like π, it cannot be written as an exact decimal or fraction.

$N = N_0 e^{-\lambda t}$
$A = A_o e^{-\lambda t}$ (and at any time, $A = \lambda N$)

Where N = number of undecayed atoms at time t
N_o = initial number of undecayed atoms
A = activity at time t
A_o = initial activity
λ = decay constant

Note that both equations have exactly the same form – which is what would be expected from the fact that A is proportional to N.

Logarithms
To do calculations with exponentials, it is necessary to use natural logarithms (written as ln on your calculator). ln is the "opposite" of e – in the same way as ×2 and ÷2 are opposites. We can therefore use ln to "cancel out" e: - eg $\ln(e^3) = 3$. NB: this **only** works if there is nothing – like a number or a minus sign – in between the ln and the e.

This idea is used to solve equations with e in them. The method is:

1. Rearrange the equation to get the part with the $e^{\text{something}}$ on its own on one side of the equation.
2. Work out the other side of the equation, so that it is a single number
3. Take ln of both sides of the equation.
4. Rearrange to find the unknown.

Worked example: Find x, given that $2 = 5e^{-0.2x}$

1. *First get the $e^{-0.2x}$ on its own, by dividing by 5:*
   $2 \div 5 = e^{-0.2x}$
2. *Work out the other side*
   $0.4 = e^{-0.2x}$
3. *Take ln of both sides:*
   $\ln 0.4 = \ln e^{-0.2x}$
   $-0.916 = -0.2x$ *(since ln and e "cancel")*
4. $x = -0.916/-0.2 = \mathbf{4.58}$

Laws of logarithms

It can be helpful to be able to simplify logarithms. To do this, use the following laws:
ln(ab) = lna + lnb
ln(a/b) = lna – lnb
$\ln(a^n) = n\ln a$
ln1 = 0

Using e and ln on your calculator

Look for e^x and lnx on your calculator. They will often be on the same button – so you'll need to press SHIFT or 2ND to access one of them.

On some calculators (including graphicals), you'll have to press e^x or lnx first, then the number, but on others, you'll have to put the number in first, then press e^x or lnx. Check which by trying to find:

e^3 (you should get 20.0)   ln2 (you should get 0.693…)

When finding something like $e^{-0.3 \times 17}$, you will need to put brackets around everything in the power – so you'd press e^x then (-0.3 × 17) (you should get the answer 0.00609…)

Relationship between decay constant and half-life

Using either the equation for activity or for number of undecayed atoms, it is possible to derive a relationship between half life ($T_{1/2}$) and the decay constant (λ).

If initial activity is A_0, then after time $T_{1/2}$, the activity will be $\frac{1}{2}A_0$.
So using the equation for activity, we get:

$$\tfrac{1}{2}A_0 = A_0 e^{-\lambda T_{1/2}}$$

Dividing both sides by A_0 gives:

$$\tfrac{1}{2} = e^{-\lambda T_{1/2}}$$

Taking logarithms gives:

$$\ln(\tfrac{1}{2}) = \ln(e^{-\lambda T_{1/2}})$$

"Cancelling" ln and e gives:

$$\ln(\tfrac{1}{2}) = -\lambda T_{1/2}$$

$$-0.69 = -\lambda T_{1/2}$$
$$0.69 = \lambda T_{1/2}$$

Hence:

$$T_{1/2} = \frac{0.69}{\lambda} \quad \text{or} \quad \lambda = \frac{0.69}{T_{1/2}}$$

Exam Hint: You do not need to remember the derivation of this formula, but you do need to know and be able to use the formula itself.

Calculations involving half life

Calculations may require you to:
- determine count-rates or time, given the half life
- determine half life from count rates
- determine numbers of atoms decaying or remaining
- determine half life from a suitable *linear* graph

The key approach is to concentrate on λ - if you are given the half-life, find λ first, and if you need to find the half life, first find λ, then use it to get $T_{1/2}$.

The following examples illustrate these.

Example 1.
A sample of uranium-238 contains 2.50×10^{23} atoms. Given that the half-life of uranium-238 is 1.42×10^{19} seconds, find

(a) the decay constant, λ;

(b) the initial activity of the sample, in becquerels;

(c) the number of uranium-238 atoms remaining after 3.0×10^{18} s.

(a) $\lambda = \frac{0.69}{T_{1/2}} = \frac{0.69}{1.42\times10^{19}} = 4.88 \times 10^{-20}$ *(3SF)*

(b) This requires using the equation $A = \lambda N$, since this is the only way to relate activity to number of atoms:

$A = 4.9 \times 10^{-20} \times 2.50\times10^{23} = 1.2 \times 10^4$ *Bq*

(c) Since this asks about a number of atoms, we must use $N = N_0 e^{-\lambda t}$
We have $N_o = 2.50 \times 10^{23}$; $t = 3 \times 10^{18}$

So $N = 2.50\times10^{23} \times e^{-4.9\times10^{-20}\times3\times10^{18}}$
$= 2.50 \times 10^{23} \times e^{-0.1464}$
$= 2.2 \times 10^{23}$ *atoms left*

Example 2
The activity of a radioactive source was measured as 10428 Bq on one day; at the same time on the following day its activity was 9135 Bq.

(a) Assuming the decay product of the source is stable, determine the half-life of the source

(b) Explain why it was necessary in part (a) to assume the decay product was stable

(c) Determine the time taken, from the first day, for the activity of the source to decay to 1000 Bq

(a) Since activity is referred to, we need to use $A = A_o e^{-\lambda t}$
$A_o = 10428$; $t = 24 \times 60 \times 60 = 86400$ s
$9135 = 10428e^{-86400\lambda}$

We need to get the part involving e on its own:
$9135/10428 = e^{-86400\lambda}$
$0.876 = e^{-86400\lambda}$

Now we must take logs:
$\ln 0.876 = \ln e^{-86400\lambda}$
$-0.132 = -86400\lambda$
$\lambda = 0.132 \div 86400 = 1.53 \times 10^{-6}\ s^{-1}$

But we want $T_{1/2} = \frac{0.69}{\lambda} = 4.5 \times 10^5$ *s*

(b) If the decay product had not been stable, it would have contributed to the measured count rate by emitting radiation itself.

(c) Using $A = A_o e^{-\lambda t}$:
$1000 = 10428\, e^{-1.53\times10^{-6}\, t}$
$1000/10428 = e^{-1.53\times10^{-6}\, t}$
$0.0959 = e^{-1.53\times10^{-6}\, t}$
$\ln 0.0959 = -1.53 \times 10^{-6}\, t$
$t = \ln 0.0959/(-1.53\times10^{-6}) = 1.5 \times 10^6$ *s*

Finding numbers of atoms from the mass of element

You may be told the mass of an element, and need to find the number of atoms in it. To do this:

- First divide the mass by the **nucleon number of the element**
- Then multiply the answer by N_A – Avagadro's number – which is approximately 6.02×10^{23}. (You **do not** have to remember this number – you will be given it if it is needed)

Example 3
A sample consists of 0.236g radioactive iron (^{59}Fe). Given that the half-life of this isotope is 46 days, calculate:
(a) the decay constant, λ;
(b) The initial activity of the sample, in bequerels

(a) $T_{1/2} = 46 \times 24 \times 60 \times 60 = 3.97 \times 10^6 s$

$$\lambda = \frac{0.69}{T_{1/2}} = \frac{0.69}{3.97 \times 10^6} = 1.7 \times 10^{-7} s^{-1}$$

(b) Number of atoms $= 0.236/59 \times 6.02 \times 10^{23} = 2.408 \times 10^{21}$
Initial activity $= \lambda N = 1.7 \times 10^{-7} \times 2.408 \times 10^{21} = 4.1 \times 10^{14} Bq$

Determining half-life from a linear graph
It is much easier to determine a quantity from a straight line graph instead of a curve, since quantities such as the gradient and intercept can be measured accurately, and a best straight line can be drawn reliably.

For $A = A_o e^{-\lambda t}$ (or $N = N_o e^{-\lambda t}$), we need to use logarithms to get the unknown (λ) and the variable (t) out of the power:

$$A = A_o e^{-\lambda t}$$

Take logarithms:

$$\ln A = \ln(A_o e^{-\lambda t})$$

Use the laws of logarithms:

$$\ln A = \ln A_o + \ln e^{-\lambda t}$$

Simplify:

$$\ln A = \ln A_o - \lambda t$$

By comparing this to the straight line equation "y = mx + c", we obtain:

In a graph of $\ln A$ (y-axis) against t (x-axis)
The gradient is $-\lambda$
The y-intercept is $\ln A_0$

Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

(a) At the start of an experiment, a sample contains 30.0 μg of isotope A, which has a half life of 50s, and 60.0μg of isotope B, which has a half-life of 25s. After what period of time will the sample contain equal masses of A and B?
(you may assume that the decay products of both A and B are stable) **[2]**

mass of A: 30 15
mass of B 60 30 15 ✓ 1/2

The student has clearly understood what is required, but has neglected to actually write down the time required – s/he should have read over the question to ensure it was fully answered.

(b) A student says: "If the masses of A and B are equal, their activities will be equal".
Explain carefully why this statement is not generally correct. [3]

Because their decay constants are different ✓ 1/3

Poor exam technique – one statement cannot be worth 3 marks, and the word "explain" in the question indicates that more is required.

(c) Calculate the activity of A two minutes after the start of the experiment, given that its relative atomic mass is 230
($N_A = 6.02 \times 10^{23}$) **[6]**

$\lambda = 0.69/50 = 0.014$ ✓
mass $= 30e^{-120 \times 0.014}$ ✓ $= 5.6\mu g$ ✓
Activity = mass $\times \lambda = 0.787 s^{-1}$ ✗ 3/6

The student has neglected to change mass into number of atoms – again, poor exam technique – why has N_A not been used?

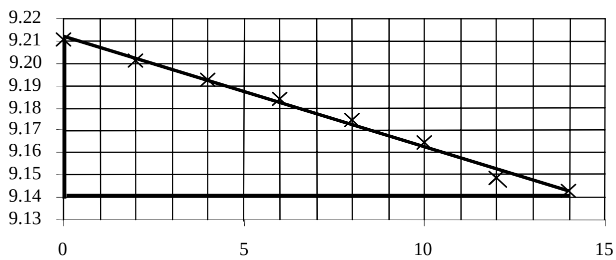
Examiner's Answers
(a) B declines to15μg in 2 half lives; A declines to 15μg in 1 half life ✓
So 50 seconds ✓
(b) Decay rate depends on number of atoms and decay constant ✓
Number of atoms will be different even if the mass is the same, because atomic mass will be different. ✓ Decay constant will be different as half lives are different. ✓
(c) $\lambda = 0.69/50 = 0.014$ ✓
mass $= 30e^{-120 \times 0.014}$ ✓ $= 5.6\mu g$ ✓
Number of atoms $= 5.6 \times 10^{-6}/230 \times 6.02 \times 10^{23}$ ✓ $= 1.5 \times 10^{16}$ ✓
Activity $= \lambda N = 0.014 \times 1.5 \times 10^{16} = 2.1 \times 10^{14}$ Bq ✓

Typical Exam Question
The table below shows how the activity of a radioactive sample reduces with time.

Time / s	Activity / s^{-1}
0	10000
2	9912
4	9825
6	9738
8	9647
10	9550
12	9400
14	9350

(a) Use the data to plot the relevant straight-line graph that will allow you to determine the half-life of the sample. [5]
(b) Use your graph to calculate the decay constant, and hence the half-life, of the sample [3]

(a) Need to plot lnA against time.
lnA values are:
9.210, 9.202, 9.193, 9.184, 9.174, 9.164, 9.148, 9.143 ✓✓



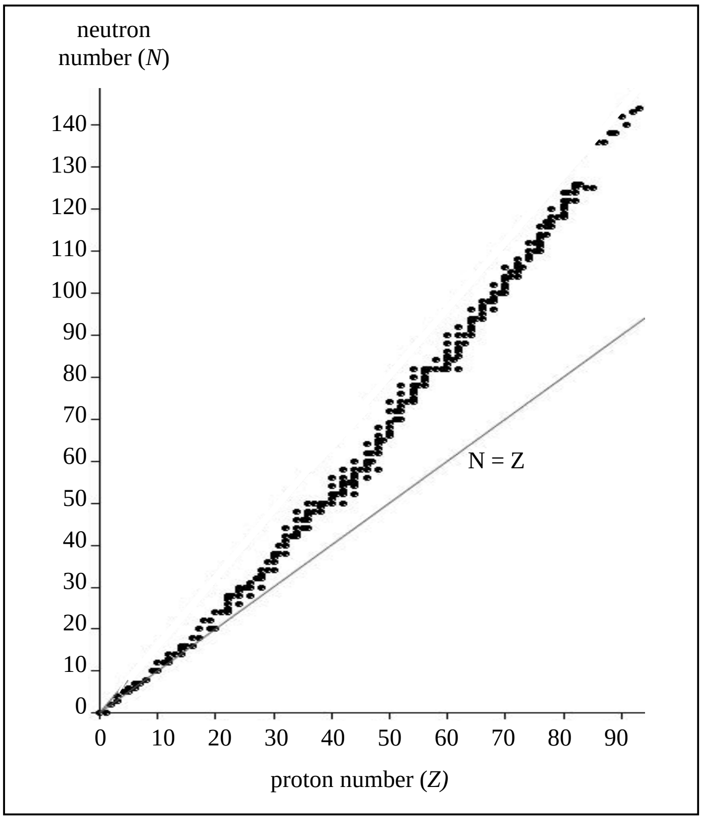
axes + labels ✓ points ✓ line ✓

(b) $\lambda = (9.21 - 9.14)/(14 - 0)$ ✓ $= 5 \times 10^{-3}$ ✓
$T_{1/2} = 0.69/\lambda = 140s$)

Which nuclei will decay?

Stability of nuclei may be shown on an **N-Z chart** (fig 1). This shows the most stable radioactive isotopes as well as the stable nuclei.

Fig 1. N-Z chart



Note that small stable nuclei have approximately the same number of neutrons and protons, but larger nuclei have an increasing proportion of neutrons. The extra neutrons moderate the effect of the electrostatic repulsion between the protons

Unstable nuclei may:
- be too large – there are no stable nuclei with Z > 83;
- have too many protons for the number of neutrons (and so be to the right of the line of stability);
- have too many neutrons for the number of protons (and so be to the left of the line of stability).

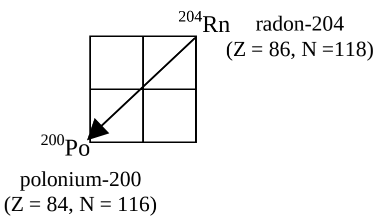
Decay modes

Radioactive decay will usually produce a daughter nucleus that is closer to the line of stability than the parent nucleus.

Radioactive decay may involve:

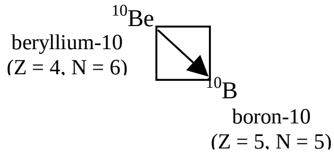
Alpha (α) decay – this removes 2 protons and 2 neutrons from the nucleus, so both *N* and *Z* decrease by 2. On the *N-Z* chart, this corresponds to a move of 2 downwards and 2 to the left

eg. radon-204 decays by α emission to give polonium-200



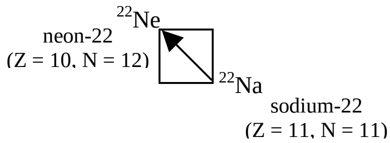
Beta-minus (β⁻) decay – this involves a neutron in the nucleus emitting an electron and changing to a proton, so *N* decreases by 1 and *Z* increases by 1. On the *N-Z* chart, this corresponds to a move downwards by 1 and to the right by 1.

eg. beryllium-10 decays by β^- emission to give boron-10



Beta-plus (β^+) decay – this involves a proton in the nucleus emitting an positron and changing to a neutron; it only occurs in man-made nuclei. This results in N increasing by 1 and Z decreasing by 1. On the N-Z chart, this corresponds to a move upward by 1 and to the left by 1.

eg. sodium-22 decays by β^+ emission to give neon-22



Since α decay is the only decay mode to reduce the total number of nucleons in the nucleus, nuclei that are simply too large to be stable will always have α decay somewhere in their decay series, although other decay modes may also be present. Nuclei that are to the left of the line of stability (i.e. neutron rich) will tend to decay via β^- emission, since this will result in products closer to the line of stability. Proton-rich small nuclei, lying to the right of the line of stability, can produce daughter nuclei closer to the line via β^+ emission; for large proton-rich nuclei, α-emission will also bring them closer to the line, since the ratio of neutrons to protons required for stability is smaller for lower atomic mass.

Questions

1. Explain what is meant by the *decay constant*, and state the relationship between the decay constant and half life.
2. Write down the equation for activity at time t.
3. Explain how to find the half-life of a radioisotope from readings of its activity at 10 second intervals.
4. Radon-224 has a half-life of 55 seconds.
   (a) Calculate the activity of a sample of 0.4g radon–224
   (b) Find the time required for there to be 0.06g radon-224 remaining in the sample.
5. A sample of magnesium-28 has an activity of 1.58×10^8 Bq. Ten hours later, its activity has fallen to 1.13×10^8 Bq. Calculate the half-life of magnesium-28.
6. A radioactive isotope of strontium, $^{90}_{38}\text{Sr}$, decays by β^- emission to form an isotope of yttrium (Y).
   (a) Write an equation representing this process
   (b) Would you expect $^{90}_{38}\text{Sr}$ to lie to the left or the right of the line of stability on an N-Z chart? Explain your reasoning.

Answers

Answers to 1. – 3. can be found in the text

4. (a) $\lambda = 0.69/55 = 0.013\ \text{s}^{-1}$
   No of radon atoms $= 0.4/224 \times 6.02\times10^{23} = 1.1 \times 10^{21}$
   Activity $= 0.013 \times 1.1 \times 10^{21} = 1.4 \times 10^{19}$ Bq
   (b) $0.06 = 0.4\ e^{-0.013t} \Rightarrow \ln 0.15 = -0.0126t \Rightarrow t = 150\text{s}$
5. Working in hours: $1.13 \times 10^8 = 1.58 \times 10^8 \times e^{-\lambda 10}$
   $\ln(1.13 \times 10^8/(1.58 \times 10^8)) = -\lambda 10 \Rightarrow -10\lambda = -0.335 \Rightarrow$
   $\lambda = 3.35 \times 10^{-2}\ \text{hours}^{-1} \Rightarrow T_{1/2} = 0.69/\lambda = 20.6$ hours
6. (a) $^{90}_{38}\text{Sr} \rightarrow {}^{90}_{39}\text{Y} + {}^{0}_{-1}\text{e}$
   (b) To the left. Since it decays by β^- emission, it will be neutron-rich compared to stable isotopes.

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Physics Factsheet

www.curriculum-press.co.uk Number 85

A2 Questions Radioactivity

Radioactive decay lends itself to a broad range of assessment. In addition to practical investigations, problems can be set to test theory, calculations, graphical work, and health and environmental issues. Written answers in paragraph form can be demanded (although they still tend to be marked in terms of separate points made).

Some of the topics that can be assessed include:

1. Types and properties of decay products (including scattering theory and the nuclear atom – Rutherford).
2. The decay process (including sources of radiation [man-made and natural], equations of decay, N-Z graphs, decay chains, etc.).
3. Decay graphs (including ideas of activity, decay constant, half-life, background correction, etc.)
4. Maths based on $A=\lambda N$, $\lambda T_{1/2} = 0.693$, and equations of the form $A=A_0e^{-\lambda t}$.
5. The inverse square law for gamma radiation.
6. Environmental issues (safety/waste disposal/half-lives).
7. Medical uses and dangers (this topic has already been covered in some depth in Factsheets 77 and 82).

This Factsheet will deal almost exclusively with answering questions on these topics. There are four exam-style questions to illustrate various points and demonstrate the methods of solution required. Then there are a number of short questions for you to attempt – the solutions are supplied.

Exam-style question 1

1 (a) Discuss problems involved in safe storage of radioactive waste (3)

(b) Complete this table relating the range, ionising power, and dangers associated with decay products α, β, and γ.

Decay Product	Range in Air	Ionising Power	Danger to humans
α	Very small	(1)	(2)
β	(1)	Medium (1)	Most dangerous if taken into body in breathing or as food / drink. Less dangerous from external source.
γ	(1)		(2)

> *Exam Hint: Be clear about the link between range and ionising power for α, β, and γ decay, and the differences between these decay products.*

Exam-style question 1 solutions

1 (a) half-life very long (1)
possible leakage into environment / water supply (1)
geological instability causing disruption to security of storage (1)
radioactivity may degrade storage vessel (1)
-any three or these or other sensible points.

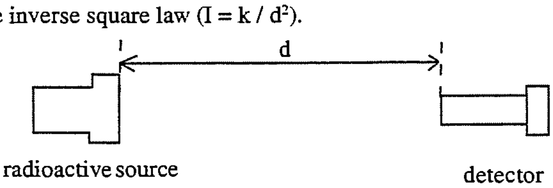
> *Exam Hint: Be prepared to write essay-type answers concerning safety and environmental issues. Remember, the number and quality of points you make are important – not the total number of words you write.*

1(b)

Decay Product	Range in Air	Ionising Power	Danger to humans
α	Very small	Very large (1)	Most dangerous if taken into body in breathing or as food / drink. (1) Less dangerous from external source. (1)
β	Small (1)	Medium	Most dangerous if taken into body in breathing or as food / drink. Less dangerous from external source.
γ	Very large (1)	Very small (1)	Most travel straight through body (1) Equally dangerous externally or internally (1)

Exam-style question 2

2. An experiment is set up to confirm that gamma-ray intensity obeys the inverse square law ($I = k / d^2$).



(a) Why will this experiment not give the required result for α and β particles? (1)

(b) How can you ensure that alpha or beta particles from the source cannot reach the detector? (1)

(c) Should the source have a long or a short half-life? Explain your answer. (2)

(d) The count rate is taken over a range of values for distance **d**. Explain what correction must be made to the recorded count rate in investigations of this sort, and how the final corrected value is determined. (2)

(e) These values for corrected count rate (s^{-1}) at various distances are found. Show graphically that the results obey the inverse square law. (7)

d /m	count rate /s		
0.20	442		
0.40	112		
0.60	51		
0.80	28		
1.00	18		
1.20	12		

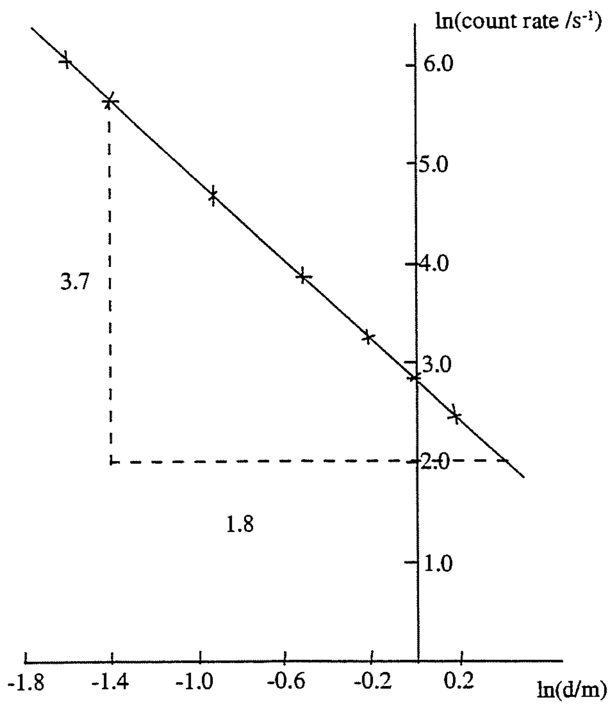
(The third and fourth columns are for your use.)

Exam-style question 2 solutions

2. (a) some (if not all) particles absorbed by air (1)
   (b) metal filter (sheet) in front of source to absorb them (1)
   (c) long half-life (1)
   to ensure constant (ignoring randomness of decay) activity of source (1)
   (d) correct for background radiation (1)
   measure background and subtract from recorded values (1)
   (e) count rate = k/d^2 , ln(count rate) = ln k – 2 ln d (1)
   the logarithmic graph should be a straight line with gradient –2 (1)

d /m	count rate /s	ln (d /m)	ln count rate /s
0.20	442	-1.61	6.09
0.40	112	-0.92	4.72
0.60	51	-0.51	3.93
0.80	28	-0.22	3.33
1.00	18	0.00	2.89
1.20	12	0.18	2.48

(2 marks for table)



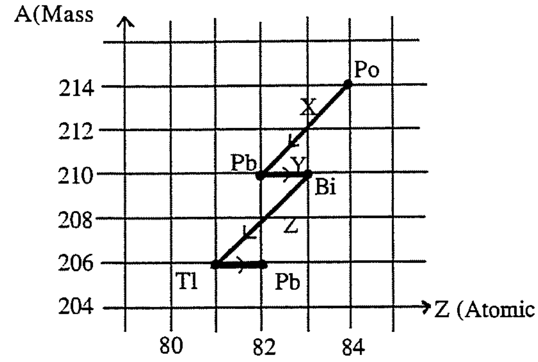
(2 marks for graph)

Gradient = -3.7 / 1.8 = -2.1 (1)

Alternatively, the marks could be gained by calculating the values for d^2, and plotting count rate against $1 / d^2$, and obtaining a straight line through the origin.

Exam-style question 3

This is a small section of the Uranium-238 decay chain:



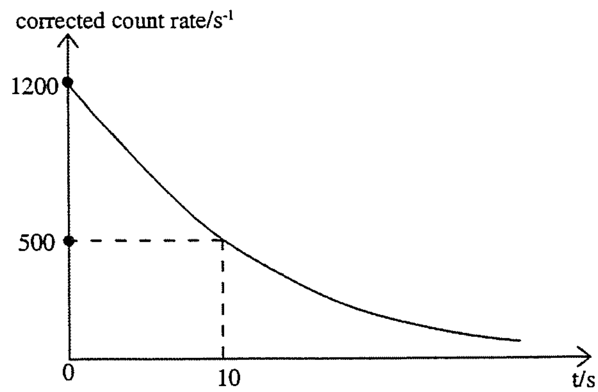
(a) Identify the decay products in sections X, Y, and Z of the chain. (1)
(b) Could the decay from Po-214 to Pb-206 be accomplished in less than four steps? (1)
(c) What information does this chart give us about gamma ray emission? Explain your answer.(2)
(d) Why would the reverse process (decay from Pb to Po) be impossible? (1)

Exam-style question 3 solutions

(a) X = α, Y = β, Z=α (1)
(b) No. (but there could be other pathways possible) (1)
(c) No information about γ rays. (1) They don't appear in the chart, as they don't result in a change in nuclear charge or mass. (1)
(d) Each decay process results in the loss of mass and/or energy from the nucleus. The reverse process would mean mass and/or energy would have to be gained in each step. (1)

Exam-style question 4

The sketch graph shows the corrected count rate measured for the decay of a radioactive sample (the sample-detector distance is constant):



(a) At time t=0s, what factors will affect the count-rate measured? (3)
(b) If the count-rate is $1200s^{-1}$ at t=0s, and is $500s^{-1}$ after 10s, find the decay constant for the radioactive isotope. (2)
(c) What is the half-life of this isotope? (1)
(d) If the measured count-rate is only 1% of the activity of the sample, find the number of atoms of this radioisotope present after 10s. (2)

Exam-style question 4 solutions:

(a) the decay constant (or half-life) of the radioactive isotope
the number of atoms **of this radioisotope** present
the distance **d**
the type or size of the detector
the type of decay product (α, β, or γ)
-any 3 of these points, or sensible alternatives
(b) count rate ∝ activity
$A = A_0 e^{-\lambda t}$, $500 / 1200 = e^{-10\lambda}$, $\lambda = 0.088\ s^{-1}$. (2)
(c) $T_{½} = 0.693 / \lambda = 7.9$ s (1)
(d) $A = 500 \times 100 = 5.0 \times 10^4$ (1)
$A = \lambda N$, $N = A / \lambda = 50\ 000 / 0.088 = 5.7 \times 10^5$ atoms. (1)

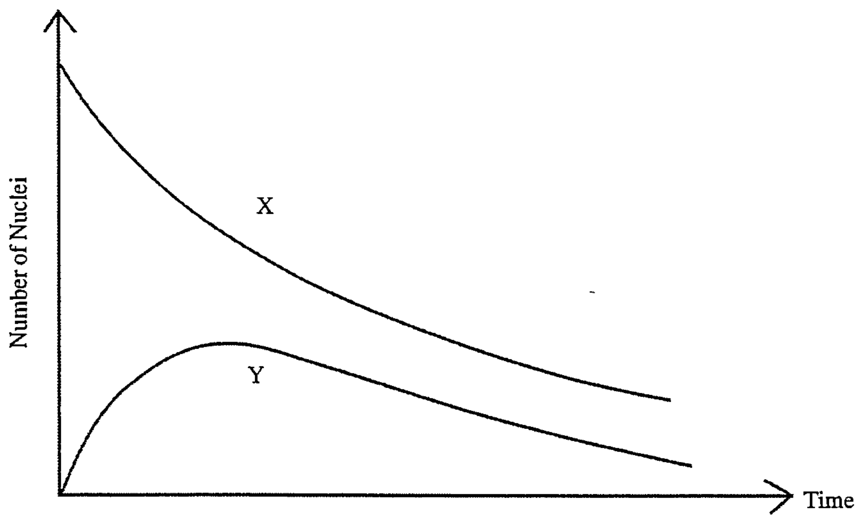
Practice Questions

1. What is the SI base unit for the rate of radioactive decay?
2. At a certain stage in a radioactive decay practical, the count rate was found to be 84Bq. If the average background measured was 254 counts in 2 minutes, find the corrected count rate (to the nearest whole number).
3. The following count rates were measured over successive 10s intervals in a beta decay practical. The source-detector distance was fixed, and the source half-life was 26 years.
   162, 188, 163, 152, 183, 171, 159, 173

   (a) Find the average count rate in Bq.
   (b) Find the maximum percentage error in these results.
   (c) What factors were responsible for the variation in readings.
4. Explain in words why the decay constant, λ, is inversely proportional to the half-life, $T_{1/2}$.
5. (a) The half-life of a source is 22 years. Find the decay constant, λ.
   (b) The decay constant for a radioisotope is $1.4 \times 10^{-4}\ s^{-1}$. Find the half life, $T_{1/2}$.
6. Complete the following decay equation:

   $${}^{y}_{x}A \longrightarrow {}^{y-8}_{x-3}B + \ ? \ + \ ? \ + \gamma$$

7. Here is a sketch graph of simultaneous decay from radioisotopes X and Y:



Explain what is happening.

Answers

1. $Bq = s^{-1}$
2. $254 / (2 \times 60) = 2.1$, $84 - 2 = 82$Bq
3. (a) 169 in 10s = 16.9Bq
   (b) (188-169) / 169 = 0.112 = 11.2%.
   (c) randomness of decay from source and **randomness** of background (not just "background").
4. The decay constant is the probability of decay for each unstable nucleus in the next second. The higher the probability of decay, the shorter time for half of the nuclei to decay.
5. (a) $\lambda = 0.693 / T_{1/2} = 0.693 / (22 \times 365 \times 24 \times 3600) = 9.99 \times 10^{-10} s^{-1}$.
   (b) $T_{1/2} = 0.693 / \lambda = 0.693 / 1.4 \times 10^{-4} = 4950s = 1.38$ hours.
6. Insert diagram 6.

   $${}^{y}_{x}A \longrightarrow {}^{y-8}_{x-3}B + 2\,{}^{4}_{2}\alpha + {}^{0}_{-1}\beta + \gamma$$

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Physics Factsheet

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Radioactive Decay Calculations

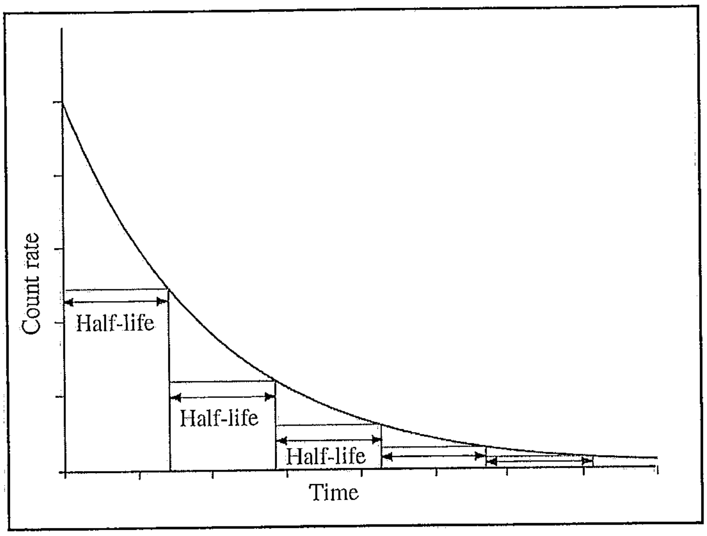
Half-life and decay constant

The half-life is the time taken for the radioactivity (A) of a substance to fall by half. The original number of radioactive nuclei (N) and mass of original substance (m) also fall by half in the same time. The decay constant, λ, is related to the half-life, $t_{1/2}$ by the equation

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

> *The time taken for radioactivity to fall by half is the half-life. Half-life and decay constant are related by the equation:*
>
> $$t_{1/2} = \frac{\ln 2}{\lambda}$$

Radioactivity over time



Exponential decay

If a radioactive substance decays into a stable one, then less radiation is emitted over time. If the original activity is A_0, then the activity at any time, A is given by $A = A_0e^{-\lambda t}$

There are similar equations for the number of radioactive nuclei $N = N_0e^{-\lambda t}$ and mass of radioactive substance $m = m_0e^{-\lambda t}$.

> $A = A_0e^{-\lambda t}$ is the relationship for the exponential decay of radioactivity.

> Count rate can be measured per second (s^{-1}) or per hour (h^{-1}) etc. The count rate per second is also measured in Becquerel (Bq).

Example 1

The nuclide $^{222}_{86}Rn$ emits alpha particles. The initial alpha particle count rate was 1000Bq. After 12 hours the count rate fell to 120Bq. Assume corrections have been made for background radiation and radioactive daughter products.

(a) Calculate the decay constant of $^{222}_{86}Rn$.
(b) Calculate the half-life of the substance.
(c) Calculate the number of $^{222}_{86}Rn$ nuclei when the count rate was 1000Bq.

Answer

(a) $A = A_0e^{-\lambda t}, \frac{A}{A_0} = e^{-\lambda t}$

$$\frac{A_0}{A} = e^{-\lambda t},\ ln\left(\frac{A_0}{A}\right) = \lambda t$$

$$\frac{ln\left(\frac{A_0}{A}\right)}{t} = \lambda$$

$$\frac{ln\left(\frac{1000}{120}\right)}{(12 \times 60 \times 60)} = \lambda = 4.91 \times 10^{-5} S^{-1}$$

(b) $t_{½} = \frac{ln2}{\lambda}$

$$t_{½} = \frac{ln2}{4.91 \times 10^{-5}} = 1.41 \times 10^{4}\ s$$

(c) $A = -\lambda N$

$$\frac{A}{\lambda} = N = \frac{1000}{4.91 \times 10^{-5}} = 2.03 \times 10^{7}$$

Example 2

The activity of a source of β radiation falls to 56% of initial value in 518s. Calculate the decay constant of the source.

Answer

$$\frac{ln\left(\frac{A_0}{A}\right)}{t} = \lambda, \frac{ln\left(\frac{A_0}{0.56A_0}\right)}{t} = \lambda, \frac{ln\left(\frac{1}{0.56}\right)}{518} = \lambda = 1.11 \times 10^{-3} s^{-1}$$

Example 3

A radioactive isotope has a half-life of 27 years. A sample has an activity of 10^6 counts s^{-1}. Calculate:
(a) the decay constant of this substance in $seconds^{-1}$.
(b) the original number of nuclei in the sample.
(c) the time taken for the activity to fall from 10^6 to 10^4 counts s^{-1}.

Answer

(a) $t_{½}$=27 years or $27 \times 365 \times 24 \times 60 \times 60 = 8.51 \times 10^8$ seconds
$\lambda = ln2/t_{½} = ln2/8.51 \times 10^8 = 8.14 \times 10^{-10} s^{-1}$
(b) $A = -\lambda N$ $A/\lambda = N = 10^6 s^{-1}/8.14 \times 10^{-10} s^{-1} = 1.23 \times 10^{15}$ particles
(c) $ln(N_0/N)/\lambda = t = ln(10^6/10^4)/8.14 \times 10^{-10} = 5.66 \times 10^9 s$ or 179 years

Exam Hint: If half-life is given in seconds, then units of decay constant will be seconds^{-1} ($t_{1/2}$ in years, λ in years^{-1}).

Example 4

If an initial radioactive count is 160Bq and background radiation is 2Bq, how many half-lives pass before the count reaches the point where background radiation is on average 10% of the count rate?

Answer

After 1 half-life, count rate is 80Bq, 2 half-lives 40Bq and 3 half-life lives 20Bq.

If the decay constant is 2.7×10^{-3}s^{-1}, how much time has passed?

$$t_{1/2} = \frac{ln2}{l}, \frac{ln2}{2.7'10^{-3}} = t_{1/2} = 257 \text{ seconds}$$

3 half lives = 771 seconds

Example 5

If there are 1.5 moles of a radioactive substance with a half-life of 2.3 years, what is the activity?

Answer

$$\frac{dN}{dt} = A = -\lambda N$$

$$\lambda = \frac{ln2}{t_{1/2}} = \frac{ln2}{2.3 \times 365 \times 24 \times 60 \times 60} = 9.56 \times 10^{-9}\ s^{-1}$$

N = nA where n is the number of moles and A is Avogadro's constant

N= 1.5 × 6.022×10^{23} = 9.033×10^{23} nuclei.

Activity = λ × N = 9.56×10^{-9}s^{-1} × 9.033×10^{23} = 8.64×10^{15}Bq

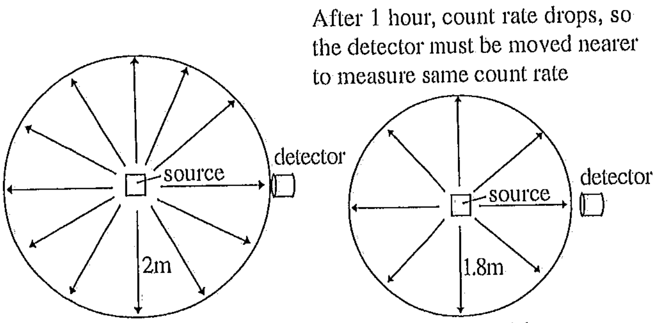
Example 6

The decay constant of a radioactive substance is 5.1×10^{-5}s^{-1} as measured by a detector 2m away. Where should the detector be moved to measure the same count rate after 1 hour?

Answer

$$A = A_0e^{-\lambda t}, \frac{A}{A_0} = e^{-\lambda t}, \frac{A}{A_0} = e^{-5.1\times10^{-5}\times60\times60} = 0.83$$

Assume the emitted radiation spreads out in all directions spherically. When it reaches the detector it all passes through a sphere of radius 2m. A small fraction of the radiation is detected as it passes through the detector.

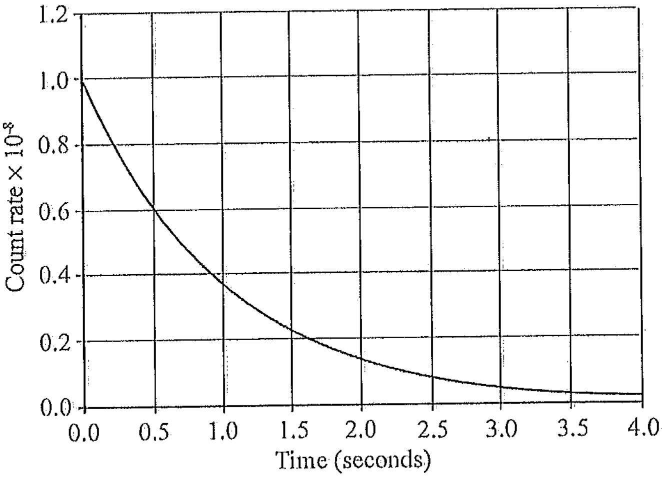


a) Time = 0 b) Time = 1 hour

After 1 hour, the count rate has dropped to 0.83 of the original rate. The area of the sphere x, that the radiation passes through at the detector must also be reduced to 0.83 of the original area to preserve the count rate. 0.83 = x^2 / 2^2 , x =1.82m

Example 7

Find the decay constant of the substance in this graph.



Answer

Half-life is 0.7s. Decay constant = $\frac{ln2}{t_{1/2}}$

Decay constant = 1.0s^{-1}

Background radiation

Radiation is emitted around us all the time, from artificial and natural sources. An average of this background radiation is measured over a period of time. However, this radiation is emitted randomly so the measured counts will vary in any short time period.

Example 8

There are 10^6 atoms of a radioactive substance with a decay constant of 7.9×10^{-5}s^{-1}. Calculate the activity of the substance. Background radiation is estimated to be 20Bq.

Assuming the background radiation varies by ±10% in a 5 second period, calculate the maximum and minimum measured counts in 5 seconds.

Answer

A= λN = 7.9×10^{-5}s^{-1} × 106 = 79Bq

Background radiation counts in 5 seconds is 100±10Bq. The radiation emitted from the source in 5 seconds is 395Bq, so the maximum and minimum measured counts will be 505Bq to 485Bq.

Example 9

A patient is injected with 1cm^3 of technetium 99 with a concentration of 0.01 moles per litre and a decay constant of 3.2 × 10^{-5} s^{-1}. Calculate the number of technetium nuclei remaining in the bloodstream after 1 day. Assume no losses through excretion.

Answer

1 litre = 1000cm^3 1cm^3 =0.001litres

0.001litres × 0.01mol litre^{-1} = 1 × 10^{-5} moles

6.022×10^{23} × 10^{-5} = 6.022 × 10^{18} particles

$$N = N_0e^{-\lambda t} = 6.022 \times 10^{18} \times e^{-3.2\times10^{-5} \times 24 \times 60 \times 60} = 3.8 \times 10^{17} \text{ nuclei}$$

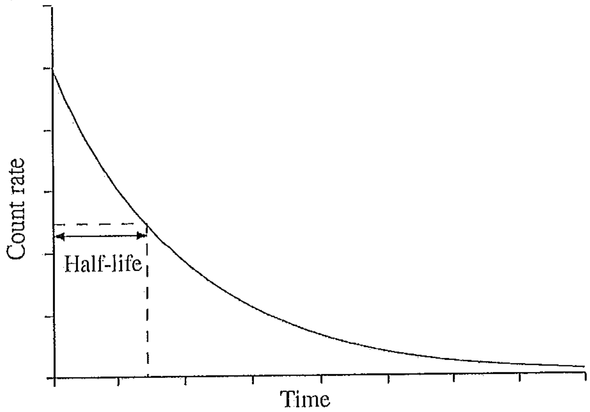
Example 10

(a) Sketch a graph of $A = A_0e^{-\lambda t}$ and show how you would identify half-life.

(b) ^{67}Cu is a beta particle emitter with a decay constant of $1.3 \times 10^{-4}\ s^{-1}$. If an initial count of 4×10^8 Bq is needed, what mass of this element is required?

Answer

(a)



(b) $A/\lambda = N = 4\times10^8/1.3\times10^{-4} = 3.1\times10^{12}$ nuclei

Mass of nuclei $= 67 \times 1.66\times10^{-27}kg \times 3.1\times10^{12}$ nuclei

$= 3.4\times10^{-13}kg$

Example 11

(a) The half-life of ^{223}Ac is 2.5 minutes. Calculate the mass needed to have an activity of 2×10^{14} Bq.

(b) The activity of a substance falls from 10^{11}Bq to 10^{10}Bq in 10 hours. Calculate the half-life.

(c) Calculate the activity of 2 mg of ^{223}Ra with a half-life of 13 hours.

Answer

(a) 2.5 minutes is 150 seconds, $\lambda = ln2/150 = 4.6\times10^{-3}s^{-1}$

$A = \lambda N$, $A/\lambda = N = 2\times10^{14}\ Bq/(4.6\times10^{-3}s^{-1}) = 4.3\times10^{16}$ *nuclei*

One nucleus of actinium has a mass of:

$223 \times 1.66\times10^{-27}kg = 3.70\times10^{-25}kg$

4.3×10^{16} *nuclei* $\times 3.7\times10^{-25}kg = 1.59\times10^{-8}kg$ *or 15.9?g*

(b) $\dfrac{ln\left(\dfrac{A_0}{A}\right)}{t} = \lambda, \dfrac{ln\left(\dfrac{10^{11}}{10^{10}}\right)}{10\times3600} = \lambda = 6.39\times10^{-5}\ s^{-1}$

$t_{½} = \dfrac{ln2}{\lambda}, \dfrac{ln2}{6.39\times10^{-8}} = 1.08\times10^4 s$

(c) $2\times10^{-6}kg$ *of* ^{223}Ra *contains*

$2\times10^{-6}kg/(223\times1.66\times10^{-27}kg) = 5.4\times10^{18}$ *nuclei*

$\lambda = ln2/t_{1/2} = ln2/(13 \times 60 \times 60) = 1.48\times10^{-5}s^{-1}$

$A = \lambda n = 5.4\times10^{18} \times 1.48\times10^{-5}s = 8.0\times10^{13}Bq$

Example 12

Calculate the activity of a source (with a half-life of 15 days) after 50 days if the initial activity is 10^8Bq.

Answer

$A = A_0e^{-\lambda t} = 10^8 \times e^{-(ln2/15\times50)} = 1\times10^7 Bq$

Practice Questions

Time (μs)	Count rate (Bq)
0.0	12.47
0.1	11.40
0.2	10.32
0.3	9.33
0.4	8.45
0.5	7.64
0.6	6.92
0.7	6.26
0.8	5.66
0.9	5.12
1.0	4.64
1.1	4.19
1.2	3.80
1.3	3.43
1.4	3.11
1.5	2.81
1.6	2.54
1.7	2.30
1.8	2.08
1.9	1.88
2.0	1.71

1. Find the decay constant of this radioactive substance by producing a graph of count rate against time.

2. (a) A source of beta radiation has an initial total activity of 2×10^7 Bq and a half-life of 24s. Calculate the activity after 1 minute.

   (b) A radiation detector has a detecting surface area of $2cm^2$. Calculate the initial detected count rate if the distance to the source of beta radiation is 0.13m.

3) (a) ${}^{40}_{19}K$ has a half life of 1.4×10^9 years and forms the stable element ${}^{40}_{18}Ar$.

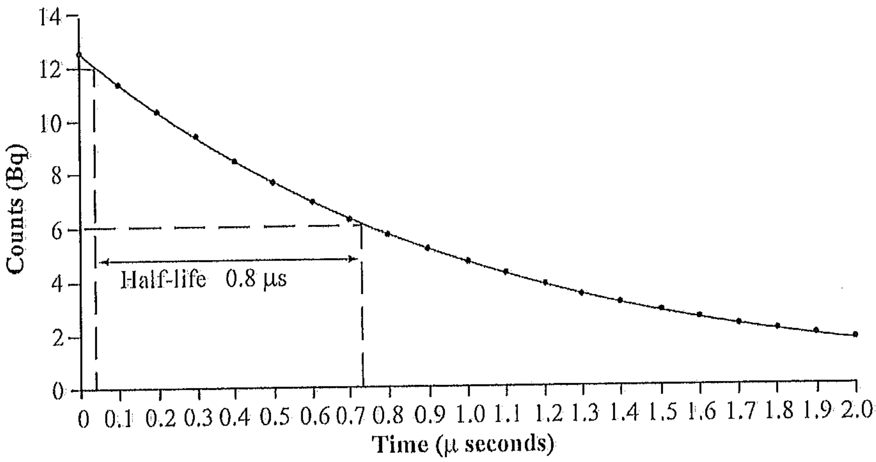
   If the ratio K:Ar is 1:3 in a meteorite, how old is the sample?

   (b) The activity of a substance falls by 37% in 14 hours. If there are initially 2×10^{22} nuclei, what is the half-life of the substance?

   (c) A source has a half-life of 120 days and an initial activity of 10^5Bq. Calculate the activity after 240 days.

Answers

1.



Half life is 0.8 μs, decay constant = ln2 / 8×10^{-7} = $8.7\times10^{5}s^{-1}$

2. (a) $A = A_0e^{-\lambda t} = 2\times10^{7}\times e^{\left(\frac{\ln 2}{24}\times 60\right)} = 3.54\times10^{6}$Bq

(b) 2×10^{7}Bq passes through a sphere with radius 0.13m and area= $4\pi\times0.13^2 = 0.212m^2$ or $2120cm^2$.
The detector receives 2/2120 of the initial count rate = 1.88×10^{4} Bq.

3. (a) Only 1/4 of the sample is still $^{40}_{19}K$ so 2 half lives have passed or 2.8×10^{9} years.

(b) If activity falls by 37%, then $A = 0.63\times A_0$, $\frac{\ln\left(\frac{A_0}{A}\right)}{t} = \lambda$, $\frac{\ln\left(\frac{1}{0.63}\right)}{14\times60\times60} = \lambda = 9.167\times10^{-6}s^{-1}$

$t_{1/2} = \ln2/\lambda = \ln2/9.167\times10^{-6}s^{-1} = 75610$ s or 21.3 hours.

(c) 240 days is two half-lives $A = A_0/2^2$ $A = 10^5/4 = 2.5\times10^{4}$ Bq

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Measuring Distances in Astronomy

We are going to look at the following methods for measuring distances to planets, stars, and galaxies:
- Kepler's third law
- Parallax
- The Hubble law
- Radar measurements

First, I want you to take a minute to think how amazing it is that we can measure the distance to *any* of these objects. After all, we can't exactly stretch out a tape measure to them.

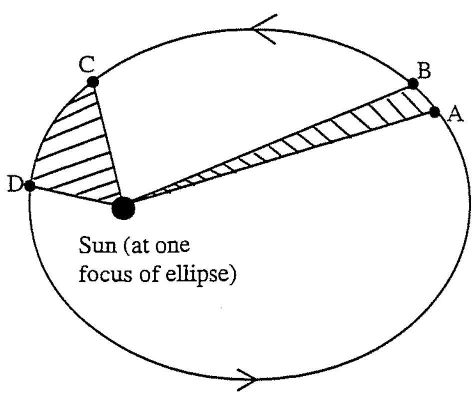
I also want you to appreciate that the difficulty in measuring astronomical distances means that many of them have a high level of experimental uncertainty. For example, the Hubble constant (see later) is often quoted with values anywhere between 50 and 80 km s^{-1} MPc^{-1}.

Kepler's third law

It might not surprise you to find that it is easier to measure the distances to planets in our Solar system, than to distant galaxies, simply because they are closer. One of the first steps to doing this was taken by Johannes Kepler, early in the 1700s. He worked out that planetary orbits obey three laws, and he worked this out purely from studying the observational data of a famous astronomer, Tycho Brahe. It is difficult to appreciate what an amazing intellectual feat this was.

Kepler's first law: Every planet orbits the Sun in an ellipse with the Sun at one focus of the ellipse.

Kepler's second law: A planet sweeps out equal areas in equal times in its orbit around the Sun.



The two shaded areas are equal. AB occurs in the same time as CD. Therefore CD faster than AB

Kepler's third law: *The square of the period of a planet's orbit is proportional to the cube of its average distance from the Sun.*

The third law is the useful one for us. (It works for any gravitational orbiting system, such as satellites around Earth, as well as planets around the Sun).

However, it has a drawback, and the drawback lies in the word "proportional". This means that we can easily find out that one planet is, say, twice as far from the Sun as another, but we need to know how far the first one was to work out the actual distance of the second (in the 18th century this was a major, major problem).

Kepler's third law can be written mathematically in three useful ways:

$T^2 \propto r^3$	where T is the **period** of orbit and r is the mean distance from the Sun
$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$	where the 1 and 2 denote two orbiting bodies.
$T^2 = r^3$	We seem to have magically changed the proportional sign to an equals sign here. The way we do this is to choose suitable units so that the constant of proportionality is '1'. $T^2 = r^3$ is true provided that **T is measured in years** and **r is measured in astronomical units (AU).**

1 AU is the mean distance from the Earth to the Sun.

Worked example

Neptune takes 165 years to orbit the Sun. How far is Neptune from the Sun?

Since we have information about the orbit in years, we can calculate a distance in Astronomical Units.

$$T^2 = r^3$$
$$165^2 = r^3$$
$$r^3 = 27225$$
$$r = 30 \text{ AU}$$ *(or 30 times further than the Earth is)*

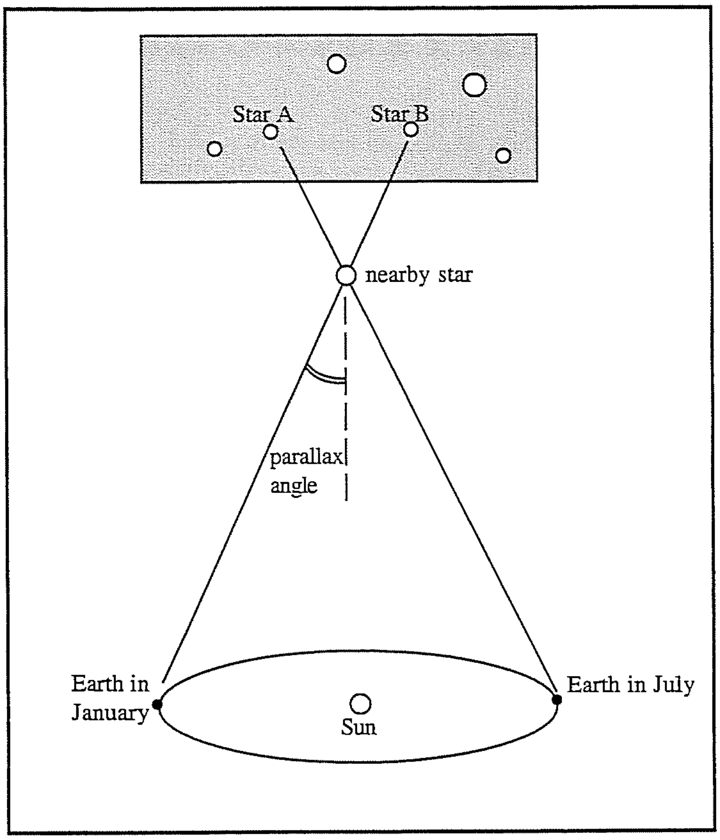
Notice that, to convert this answer into km we would have to know how far the Earth is from the Sun.

Parallax

Venturing out of our Solar system, the next objects we encounter are the nearest stars. Their distances can be calculated using the method of "**heliocentric parallax**".

When travelling on a train the foreground seems to rush past very quickly while objects near the horizon barely seem to move at all. This illusion is called **parallax** and is obviously due to the differing distances of the objects – nearby objects appear to move quicker.

The Earth is moving in orbit around the Sun, and so observing nearby objects twice, at 6 month intervals, should result in them appearing to have moved, just because we are looking at them from different angles. The following diagram should make this clearer.



- In January, the nearby star looks as though it is in front of star B.
- In July, the nearby star looks as though it is in front of star A. The closer the nearby star, the greater will be its angle of parallax. Measuring this angle (by using stars A and B), and knowing the distance from the Earth to the Sun allows us to use trigonometry to calculate the distance of the nearby star.

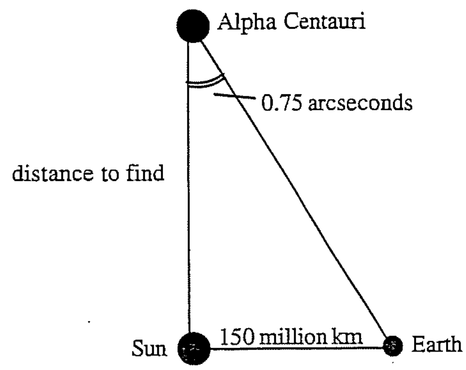
Some technical vocabulary will help at this stage:
- 1° of angle = 60 arcminutes
- 1 **arcminute** = 60 arcseconds, so
- 1 **arcsecond** = 1/3600th of a degree
- 1 **light year** (l.y.)is the distance travelled by light in a year, which is equal to 9.5×10^{15}m
- 1 **parsec** (Pc) is the distance of a star which has a ***par***allax of 1 arc***sec***ond.
- 1 Pc = 3.26 l.y.

Note that this method relies upon the background stars being far enough away that *they* don't appear to move too. In fact this is no problem, as they *are* so far away. In fact the difficulty is finding stars that are close enough to act as the "nearby star" in the diagram.

No star is close enough to have a parallax angle as large as 1 arcsecond. Even measuring 1 arc second is equivalent to distinguishing lines of writing on a page 1 km away. This means that the parallax method can only measure the distances of a very limited number of stars.

Worked example

The star Alpha Centauri has an annual parallax of 0.75



arcseconds. How far is it from the Sun in light years, if 1 AU is equal to 150 million km?

The trigonometry here is really easy, but there are still dozens of ways to go wrong, and nearly all of them involve problems with units. The diagram below shows the information in the question:

So here goes with the units:
- *0.75 arcseconds* $= 0.75 \div 3600 = (2.08 \times 10^{-4})°$
- *150 million km* $= (150{,}000{,}000 \times 1000)\ m = 1.5 \times 10^{11}\ m$
- *distance to find* $= x$

And for the trigonometry:

$$\tan(2.08 \times 10^{-4}) = \frac{1.5 \times 10^{11}}{x}$$

$$\Rightarrow x = \frac{1.5 \times 10^{11}}{\tan(2.08 \times 10^{-4})}$$

so, $x = 4.13 \times 10^{16}\ m$

and since 1 l.y. $= 9.5 \times 10^{15}\ m$,

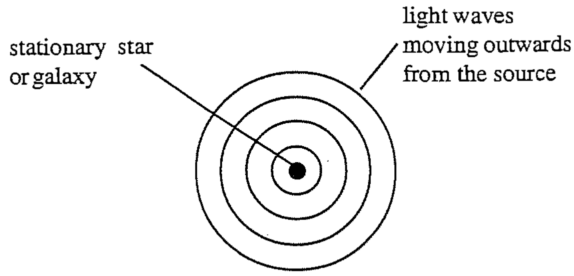
$$x = \frac{4.13 \times 10^{16}}{9.5 \times 10^{15}} = 4.3\ l.y.$$

Notice that the angle in the diagram is deliberately drawn way too large. In fact the angle is so small that the triangle is very nearly isosceles, which means that using the sine function (instead of tan) to get the *Earth*-to-star distance gives you the same answer to many significant figures. (Could you find a quicker, but approximate, and tricky to spot, way of doing this question if you were given the information that 1 Pc = 3.26 l.y.?)

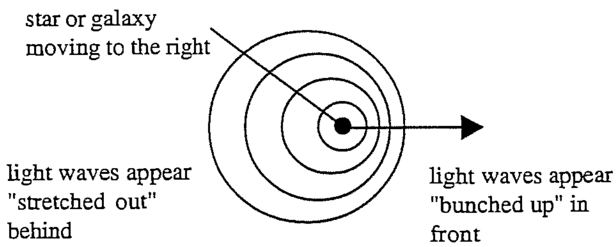
Hubble's law

I mentioned before that it is amazing we can work out how far away astronomical objects are. It is just as amazing we can work out how fast they are moving, but with stars and galaxies we can do exactly that, using the **Doppler Effect**.

Below is a diagram of wavefronts radiating from a stationary source. They could be any waves, eg sound, ripples on water, but here they are labelled as light.



Now look at what happens if the source of the waves is moving...



- If the star is coming towards you, you observe the light to have a shorter wavelength (higher frequency) than it is actually emitting. The light has been shifted towards the blue end of the spectrum – **blueshifted.**
- If the star is going away (receding) from you, you see the light to have a longer wavelength (lower frequency) than it is actually emitting. The light has been shifted towards the red end of the spectrum – **redshifted**. The redshift, Δf (or $\Delta\lambda$), is simply the difference between the *observed* frequency (or wavelength) and that which *would* be observed if the star were stationary.

The faster the star, the greater the redshift or blueshift. Provided the speed of the source of waves is much less than the speed of the waves (which is usually true if the waves in question are light waves!), then:

$$\frac{\Delta f}{f} = \frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

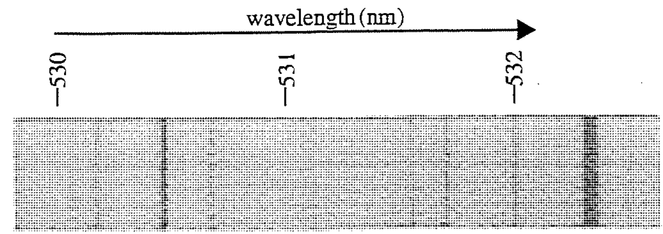
where: f is the 'true' frequency of the light
λ is the 'true' wavelength of the light
c is the speed of the waves (light)
v is the speed of the source of waves (star)

This equation is easy to understand. It simply means that if the star is travelling toward or away from us at, say, 2% of the speed of light, then we will receive light shifted in frequency and wavelength by 2%.

This is all very well, but how can you *tell* from a spectrum whether its light has been redshifted or blueshifted or neither.

Well, if the spectrum were continuous, you would not be able to, but in fact the spectrum of star is crossed by many dark lines, caused by the outer layers of stars absorbing very specific wavelengths of light as the light leaves the star. The wavelength of these lines can be measured very precisely using a **diffraction grating**. An example of a stellar spectrum is shown (Fig a).

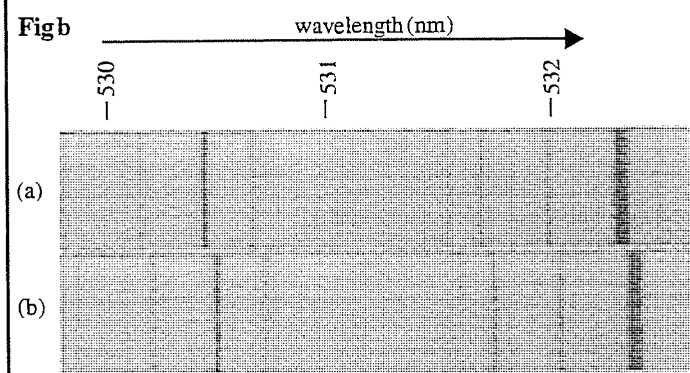
Fig a



Each chemical element has its own set of **absorption lines**. The lines in the diagram above thus act as a 'barcode' for the star, telling exactly what elements are present in the star and in what quantities.

More important, though, for distance measurement, is the fact that these absorption lines are redshifted or blueshifted according to whether the star is moving towards us or receding from us. Fig b shows a redshifted spectrum.

Fig b



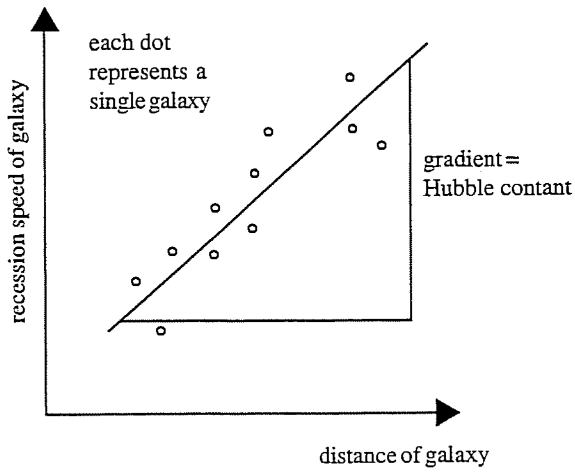
(a) spectrum of stationary star
(b) spectrum of star receding (moving away). Notice that the spectral lines have been shifted towards longer wavelengths (redshifted)

By measuring the amount of redshift/blueshift, and applying the equation above, the speed of the star can be measured (to be precise, it is the component of its velocity towards/away from us, and tells us nothing about its velocity across our line of sight).

How do we get distance measurements from this information about velocities? Well, for *stars*, we don't. But for *galaxies*, it turns out that there is a remarkable link between velocity and distance from us.

It would be reasonable to suppose that roughly half of all galaxies are moving towards us and half are receding. Edwin Hubble, in the early part of the last century, found instead that for all practical purposes

ALL galaxies have their light redshifted (and are thus receding). Further, he found that the recession speed (component of velocity away from us) is proportional to the distance from us, as in this graph.



The **recession speed**, v, of a galaxy is proportional to the distance, D, so that $v = HD$,

where H is the constant of proportionality called the **Hubble constant**. The Hubble constant is often given as **65 km s^{-1} MPc^{-1}**, but its value is uncertain because:

- the measurement of distance for each galaxy (done by other methods outside the scope of this factsheet) is subject to a large uncertainty
- the correct "line of best fit" is not obvious as the correlation is not perfect
- the gradient of the line of best fit IS the Hubble constant.

> ***Exam Hint***
> *The unit of the Hubble constant can be confusing at first sight. It does make sense however – it means that...*
> - *a galaxy 1 million parsecs away is receding from us at a speed of 65 km s^{-1},*
> - *a galaxy 2 million parsecs away is receding from us at a speed of 130 km s^{-1}, etc*
>
> *Always make sure that the units of v and H are consistent (either "km s^{-1}" and "km s^{-1} MPc^{-1}", OR "m s^{-1}" and "m s^{-1} MPc^{-1}")*

> **Worked example**
> ***A spectral line emitted by a helium-filled discharge tube has a wavelength of 590 nm when measured using a source in a laboratory on Earth. The same spectral line measured using light from another galaxy has a wavelength 634 nm. Calculate the velocity of the galaxy relative to Earth, and its distance from us.***
>
> *λ is the 'true' wavelength = 590 nm*
> *$\Delta\lambda$ is the wavelength shift = 634 nm – 590 nm = 44 nm*
>
> $$\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$
>
> $$\Rightarrow \frac{44}{590} = \frac{v}{3 \times 10^8}$$
>
> $v = 2.2 \times 10^7$ m s^{-1}
>
> $v = HD$, or $D = v/H$
>
> $$D = \frac{2.2 \times 10^4}{65}$$ notice that v has been converted into km s^{-1}. so that its units are consistent with those of H.
>
> **D = 340 MPc**

Radar measurements

Radio waves sent from the Earth can reflect from the Sun, and return to Earth. The time it takes them to do this allows us to calculate the distance to the Sun.

> ***Worked example***
> ***A radar signal is sent to the Sun and returns after 16 minutes and 40 seconds. Calculate the distance of the Sun from the Earth, if the the speed of radio waves is 3 × 10^8 m s^{-1}.***
>
> *The first thing to note is that 16 minutes and 40 seconds is (16 × 60) + 40 = 1000 s*
>
> *We can then just use 'distance = speed × time' to work out the distance the radio waves travel.*
>
> *distance = (3 × 10^8 m s^{-1}) × 1000 s*
> *= 3 × 10^{11} m*
> *= 3 × 10^8 km (300 million km)*
>
> *And since this is for the trip there and back, the distance to the Sun is __150 million km__.*

Practice Questions

For these questions you can use the following data:
Hubble constant, H = 65 km s^{-1} Mpc^{-1}
1 parsec (Pc) = 3.1×10^{16} m
1 parsec = 3.3 light years.
Speed of light in a vacuum, $c = 3.0 \times 10^8$ms^{-1}

1. A geostationary satellite takes 24 hours to orbit Earth once and is 42,000 km from the centre of the Earth. A low polar satellite has a period of orbit of 90 minutes. How far from the centre of the Earth is the low polar satellite? If the radius of the Earth is 6400 km, what are the altitudes of these two satellites?
2. A star has a parallax angle of 0.31 arcseconds. The Earth is 150 million km from the Sun. How far from Earth is it in (a) metres, (b) light years, (c) Parsecs, (d) astronomical units?
3. The frequency of a spectral line of the radiation from a galaxy is measured to be 5.634×10^{14}Hz. The same spectral line from a laboratory source has a measured frequency of 5.997×10^{14}Hz.
   Calculate:
   (a) the velocity of the galaxy with respect to the Earth,
   (b) the distance of the galaxy from Earth in MPc.
4. (a) One possible value for the Hubble constant is 65 kms^{-1}Mpc^{-1}. Calculate, in Mpc, the distance from the Earth for a galaxy travelling at 1/20th of the speed of light.
   (b) Calculate the time taken, in years, for light to travel from the galaxy in part (a) to Earth.
5. A galaxy is 5.5×10^{24} m from the Earth.
   (a) Calculate the speed of this galaxy relative to the Earth.
   (b) The galaxy emits light of wavelength 589.0 nm as it moves away from the Earth. This light is observed on the Earth.
      (i) Calculate the change in wavelength of this light due to the movement of the galaxy.
      (ii) Calculate the wavelength of the light from the galaxy when observed on the Earth.

Answers

1. 6600 km
   200 km and 35,600 km for the altitudes
2. (a) 1.0×10^{17} m (b) 10.6 l.y. (c) 3.24 Pc (d) 670,000 AU
3. (a) 1.8×10^7 m s^{-1} (b) 280 MPc
4. (a) 230 MPc (b) 750 million years
5. (a) 11500 km s^{-1}, (b) (i) 23 nm (ii) 612 nm

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Stars: Luminosity & Surface Temperature

This Factsheet explains the nature of blackbody radiation and how it relates to the surface temperature and luminosity of stars.

Blackbodies

A blackbody is a perfect absorber of all EM radiation

Since a black body absorbs all EM radiation, it reflects none of it – so you cannot see it by reflected light, which is the way we see everyday objects around us.

Physical objects that we see around us are not blackbodies – we would not be able to see them if they were! However, you can make a good approximation to a blackbody by making a small hole in a hollow sphere. Almost all radiation entering the sphere will be reflected around the inside until it is absorbed.

Kirchoff's law tells us that a body that a good absorber of EM radiation will also be a good radiator.

A blackbody can emit EM radiation at all wavelengths

The amount of radiation of each wavelength emitted by a blackbody depends only on its temperature.

This is **not** true for other bodies - they may be particularly good absorbers (and emitters) of particular wavelengths - so the energy emitted will depend on the surface.

Electromagnetic Radiation

The energy of each photon of EM radiation depends on its frequency:

$E = hf$ — *E = energy (joules)*, *h = Planck's constant*, *f = frequency (Hz)*

The energy can also be expressed in terms of the wavelength of the EM radiation, using the equation

$c = f\lambda$ c = speed of light (ms^{-1}) λ = wavelength (m)

From this equation, we obtain

$$f = \frac{c}{\lambda}$$

Substituting into the energy equation gives:

$$E = \frac{hc}{\lambda}$$

E = energy (joules), *h = Planck's constant*, *λ = wavelength (m)*

So, the shorter the wavelength of the EM radiation, the higher its energy - short wavelength, high frequency radiation like x-rays and γ-rays have the highest energy, and long wavelength, low frequency radiation like radio waves have the lowest energy.

Blackbody Radiation

As we would expect from everyday experience, the amount of energy radiated by a body depends on its temperature - a hot object will radiate more than a cooler one! The mathematical relationship between energy radiated per second and temperature is given by the Stefan-Boltzmann law (also known as Stefan's Law):

Stefan-Boltzmann law

$$L = \sigma A T^4$$

L = energy radiated per second, *σ = Stefan-Boltzmann constant*, *A = surface area (m^2)*, *T = absolute temperature (K)*

A blackbody does not radiate equally at all wavelengths - there is always a **peak wavelength** at which the radiation has the highest intensity. This is given by **Wien's Law**.

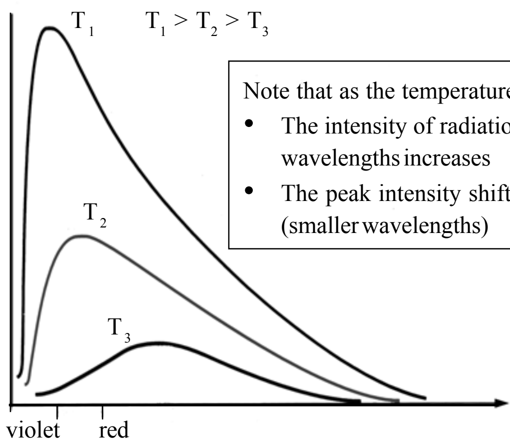
Wien's Law

$$\lambda_{max} = \frac{2.90 \times 10^{-3}}{T}$$

λ_{max} = wavelength at which radiation has maximum intensity, *T = absolute temperature (K)*

So as the temperature increases, λ_{max} decreases - the peak shifts towards shorter wavelengths (and hence higher frequencies).

The curves showing the intensity of blackbody radiation have a characteristic shape.



Note that as the temperature increases
- The intensity of radiation at all wavelengths increases
- The peak intensity shifts to the left (smaller wavelengths)

These curves also explain what we observe from a radiating body:

- At low temperatures, the peak is in infra-red (or longer) wavelengths. The intensity in the visible range is too small to be noticeable. We do not see any light emitted from the body.
- As the temperature increases, the peak wavelength moves towards the visible range. We see the body glowing red (as red has the longest visible wavelength).
- With further increases, the peak wavelength moves to orange, then yellow wavelengths.
- As the peak moves towards the centre of the visible range, with the peak in the green wavelengths, the high intensity across the visible spectrum makes the body appear to glow white-hot
- At higher temperatures still, the peak moves towards the blue end of the spectrum.

Luminosity

Luminosity (L) = energy (J) radiated per second

The units of luminosity are Js^{-1}, or ***watts***

The luminosity of a star is commonly expressed by comparing it to the luminosity of the sun ($L_\odot = 3.83 \times 10^{26}$ W). This tells us how many times brighter (or fainter) than the sun this star would appear, if it was the same distance from us as the sun.

Luminosity and Temperature

Although it seems odd, stars are actually black bodies to a good approximation. If you shine a light at the sun, it will not be reflected. But of course the sun does not look black! This is because it is radiating.

The luminosity of a star can therefore be related to its temperature as described in the Stefan-Boltzmann law. Since stars are approximately spherical, we know that $A = 4\pi r^2$, where r is the radius of the star.

For a spherical body

$L = 4\pi\sigma r^2T^4$

L = luminosity
σ = Stefan-Boltzmann constant
r = radius (m)
T = absolute temperature (K)

Remember: *The temperature referred to here is the* ***surface temperature*** *of the star. Its core temperature will not be the same.*

Example. The luminosity of the sun is 3.83×10^{26}W. Its radius is 6.96×10^8m. Calculate the surface temperature of the sun. ($\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4})

$$L = 4\pi\sigma r^2T^4$$
$$3.83 \times 10^{26} = 4\pi(5.67\times 10^{-8})(6.96 \times 10^8)^2T^4$$

$$T^4 = \frac{3.83\times10^{26}}{4\pi(5.67\times10^{-8})(6.96\times10^{8})^2}$$

$$T = \sqrt[4]{\frac{3.83\times10^{26}}{4\pi(5.67\times10^{-8})(6.96\times10^{8})^2}}$$

T = 5770 K (3SF)

There is a great variation in luminosities and surface temperatures. as shown in the table below.

Star	luminosity/$L_\odot$	Surface Temperature/K
Orionis C	30 000	33 000
Spica	8 300	22 000
Rigel	130	12 500
Altair	24	8 700
Procyon A	4.00	6 400
Alpha Centauri A	1.45	.5 900
The Sun	**1.00**	**5 800**
Tau Ceti	0.44	5 300
Pollux	0.36	5 100
Epsilon Eridani	0.28	4 830
Ross 128	0.0005	3 200
Wolf 359	0.0002	3 000

Wien's Law and Temperature

Wien's law enables us to deduce the temperature of a star from the wavelength at which it radiates with the maximum intensity. It also allows us to relate the colours of stars to their temperature.

Increasing Temperature →

red yellow-orange yellow yellow-white white blue-white blue

← Increasing Wavelength

Example. Rigel is a blue star, whose peak emission wavelength is 2.32×10^{-7}m. Use Wien's law to calculate the surface temperature of Rigel.

$$\lambda_{max} = \frac{2.90\times10^{-3}}{T}$$

$$2.32\times10^{-7} = \frac{2.90\times10^{-3}}{T}$$

$$T = \frac{2.90\times10^{-3}}{2.32\times10^{-7}} = 12500K$$

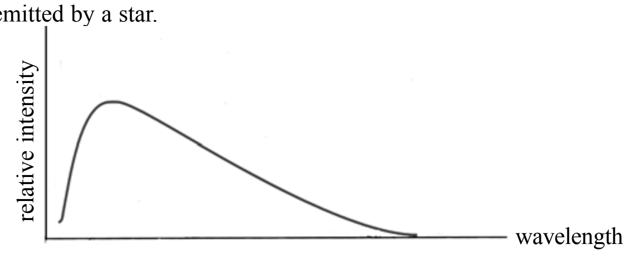
Exam Hint: *Questions on this topic will typically ask you to:*

- *Sketch a second blackbody curve on the graph you are given, for a body of different temperature*
- *Use Wien's law to estimate temperature or peak wavelength*
- *Use the Stefan-Boltzmann law to find radius, luminosity or temperature*
- *Use any of the above in conjunction with the Hertzsprung-Russell diagram (Factsheet 41)*

Questions

1. The star Becrux has a luminosity of 6.128×10^{30} W and a surface temperature of 30 000K.
Calculate an estimate for its radius *($\sigma = 5.67\times 10^{-8}$ W m^{-2} K^{-4})*

2. The graph below shows the relative intensity of different wavelengths emitted by a star.



(a) Given the star has a surface temperature of 4200K, calculate the wavelength at which the intensity is maximum.
(b) Suggest the colour of this star
($\lambda_{red} = 7\times 10^{-7}$m; $\lambda_{blue} = 4\times 10^{-7}$m)
(c) Another star has a surface temperature of approximately 6000K On the graph above, sketch a curve to show the relative intensity of wavelengths emitted by this second star.

Answers

1. $r^2 = 6.128 \times 10^{30}/(4\pi \times 5.67\times 10^{-8} \times 30000^4) = 1.062 \times 10^{19}$ $r = 3.3 \times 10^9$m
2 (a) $\lambda_{max} = 2.90\times 10^{-3}/4\,200 = 6.90\times 10^{-7}$ m
(b) red
(c)

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Physics Factsheet

September 2002 **Number 41**

Lives of Stars

Classification of stars
The classification of stars depends on their brightness or intensity and their temperature.

Magnitude
The magnitude of a star is a measure of how bright it is.

The **apparent magnitude** measures how bright the star appears from earth. However, this is not very useful, because a close, dim star and a more distant, bright star may well appear similar.

Instead, **absolute magnitude** is used.

> ***Absolute magnitude of a star is the apparent magnitude it would have if it was 10 parsecs from earth.***

Absolute magnitude takes values between –10 and + 15.
- **negative** numbers corresponding to **bright** stars
- **positive** numbers correspond to **dim** stars

So, for example, a star of magnitude –6 is brighter than one of magnitude –1, which again is brighter than one of magnitude 6.

Note that doubling the magnitude does **not** mean that the intensity is doubled or halved. A difference of magnitude of 1 corresponds approximately to a factor of 2.5 in the intensity of light.

So light from a star of magnitude –6 is 2.5 times as intense as light from a star of magnitude –5. Similarly, a star of magnitude –6 is $2.5^5 = 97$ times as bright as one of magnitude –1.

This means that magnitude is similar to a **logarithmic** scale (see Factsheet 10 - Exponentials and Logarithms).

Temperature
The (surface) temperature of a star can be determined from the colour of the light it emits. This is referred to as classification by **spectral type**.

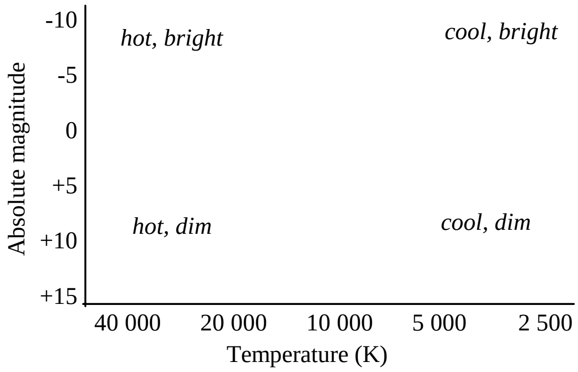
Spectral Type	Colour	Temperature
O	Blue	Hottest
B	Blue-white	↑ Increasing temperature
A	White	
F	Yellow-white	
G	Yellow	
K	Yellow-orange	
M	Red	Coolest

Why does the colour change with temperature?
Hot bodies emit electromagnetic radiation of a range of different wavelengths. There is always a peak intensity at one particular wavelength. As the body gets hotter:
- The intensity of radiation emitted at each wavelength increases
- The wavelength at which the peak occurs decreases

The hottest stars have peaks at the blue (short wavelength) end of the visible spectrum, and the coolest ones at the red (long wavelength) end.

> **Astronomical Distances**
> Distances between stars may be measured in:
>
> - **Astronomical units (AU) –** this is the mean distance between the earth and the sun.
>   1 AU $\approx 1.5 \times 10^{11}$m
> - **Parsecs (pc)** - this unit of distance is derived from the *parallax* method of measuring stellar distances.
>   1 pc = 2×10^5 AU
> - **Light-years** – this is the distance light travels in one year.
>   The speed of light is 3.00×10^8 ms^{-1}.
>   There are $60 \times 60 \times 24 \times 365$ seconds in a year.
>   So 1 light year $= 3.00 \times 10^8 \times 60 \times 60 \times 24 \times 365$
> $= 9.46 \times 10^{15}$m

Hertzsprung-Russell (H-R) Diagram
This shows the absolute magnitude of stars (y-axis) plotted against their temperature (x-axis). The scales are somewhat "unusual", in that:
- Negative magnitudes (and hence bright stars) are at the top
- High temperatures are to the left

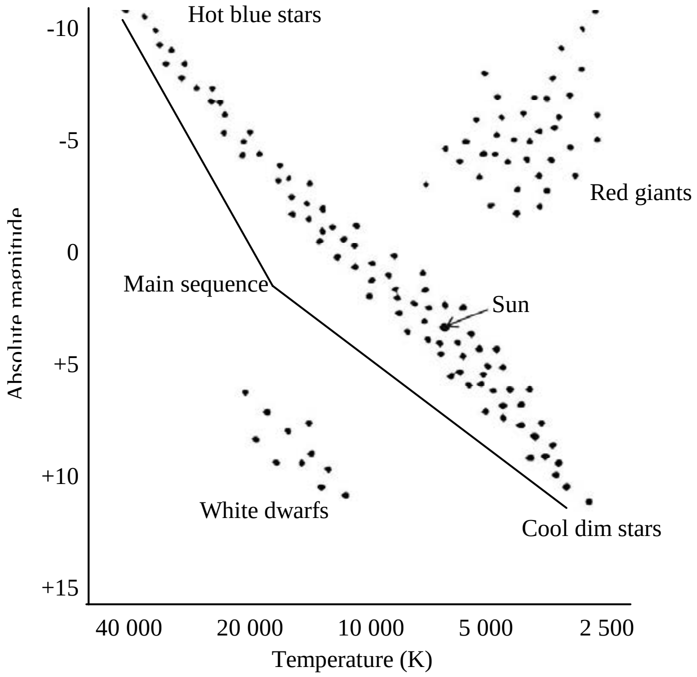


Note that the temperature scale is logarithmic – the temperature halves each horizontal unit. This is necessary to display the wide range of temperatures on one graph.

You may encounter slightly different versions of this diagram:
- The y-axis may be intensity, rather than absolute magnitude – in which case a logarithmic scale is used.
- The x-axis may give the star's spectral type (O,B,A etc)
- Both axes may have scales relative to the Sun
    - the y-axis would show intensity relative to the sun (a value of 2 indicating a star twice as intense as the sun)
    - the x-axis would show temperature relative to the sun (a value of 2 indicating a star twice as hot as the sun).

When the absolute magnitudes and temperatures of stars are plotted on an H-R diagram, we find that most of the stars fall on a diagonal band from the top left to bottom right of the diagram. These are known as **main sequence** stars.

There are two other clusters of stars on the H-R diagram:-
- A group of cool, red stars that are bright. Since they are cool, they must be very large to be so bright (typically their diameters are 10-100 times that of the sun). These are known as **red giants**.
- A group of hot, white stars that are dim. Since they are hot, they must be small to be so dim (typically their diameters are about 1% that of the sun). These are known as **white dwarfs**.



Masses of stars
Binary star systems (in which two stars orbit each other) can be used to derive estimates of the mass of stars, using data on the distance between the stars, the period of their orbit and Kepler's law. This has shown that there is a relationship between a star's position on the H-R diagram and its mass.

> ***The most massive stars are the hot, bright stars at the top left of the H-R diagram. The least massive stars are the cool, dim stars at the bottom right of the H-R diagram.***

Life of a star
The life of a star may be divided into four sections:- protostar, pre-main sequence, main sequence and post-main sequence. The end of the star's life will be considered separately.

1. Protostar
Space contains very thinly dispersed gas – largely hydrogen (H) with some helium (He). In some regions, this gas is rather more dense – these are known as **interstellar gas clouds.**

- In these clouds, random fluctuations in density will result in there being some regions where the atoms are closer together than average. Gravitational attraction between these atoms will then result in the density increasing further, so that atoms from the surrounding areas are also attracted. If this region becomes large enough and dense enough, it may produce a **proto-star**.
- As the atoms in a proto-star become closer together, they lose gravitational potential energy. This loss is balanced by an increase in kinetic energy of the atoms – which is equivalent to a rise in temperature. While the density of the cloud is sufficiently low for infrared radiation to pass through it, this thermal energy is radiated.

2. Pre-main sequence
- As the gravitational collapse of a proto-star continues, it becomes opaque to infrared radiation – the thermal energy is trapped and the star starts to heat up.
- The heating within the proto-star increases its internal pressure – this resists the gravitational contraction.
- The heat generated causes the proto-star to radiate heat and light very weakly – it has become a **pre-main sequence star**.

3. Main sequence
- The pre-main sequence star continues to increase in temperature. Eventually, the temperature becomes sufficient (at several million kelvin) to start thermonuclear fusion of hydrogen (see box). This is known as **hydrogen burning**. It has then become a **main sequence star**.
- If the initial protostar was too small (less than about 8% of the mass of the sun) it will not be massive enough to produce temperatures sufficient for fusion – it will never become a main sequence star.

> ***Exam Hint:-*** *Hydrogen burning in a star does not involve burning in the normal sense of the word – it refers to thermonuclear fusion of hydrogen, not a chemical reaction.*

Eventually, the star reaches an equilibrium state in which:
- The energy radiated by the star balances the energy produced in thermonuclear fusion – so the **temperature** is **constant.**
- The radiation pressure outwards from the core of the star balances the gravitational pressure tending to collapse the star – so the **size** is **constant**.

Hydrogen burning
Hydrogen burning occurs in three stages:-
- Two protons fuse to form a deuterium nucleus, a neutrino and a positron.
  $^1_1H + ^1_1H \rightarrow ^2_1H + \nu + ^0_1e$
- The deuterium nucleus fuses with another proton to form a helium-3 nucleus.
  $^1_1H + ^2_1H \rightarrow ^3_2He$
- Two helium-3 nuclei fuse to form a helium-4 nucleus and two protons
  $^3_2He + ^3_2He \rightarrow ^4_2He + 2^1_1H$

4. Post-main sequence
- Once the hydrogen in the core of a star is used up, the state of equilibrium is broken – it then leaves the main sequence. Some hydrogen burning may continue in a shell surrounding the core. The core itself will consist mainly of helium.
- The core then contracts, reducing its gravitational energy and increasing its temperature. This increases the rate of energy output from the core.
- The increase in energy output causes the star to expand. As it expands, its outer layers cool. A **red giant** is produced.
- The core of the red giant continues contracting, increasing the temperature still further. Eventually, the temperature is sufficient to produce fusion reactions of the helium in the core – **helium burning** – which produces beryllium, carbon and oxygen nuclei.
- Helium burning maintains the red giant in a stable state for 10-20% of the time it spent on the main sequence.

Death of the star

Eventually the helium is exhausted in the core of a red giant; its core then collapses again. What happens next depends on the size of the star.

(a) Mass of core less than 1.4 solar masses
(main sequence mass below 5 solar masses)

- No further thermonuclear reactions are started, as the temperature is not sufficient
- The star sheds its outer layers of gas – approximately 50% of its mass. These form a **planetary nebula**
- As there is no nuclear burning, there is no outward pressure to balance gravity – the core is compressed further.
- This collapse is halted when electrons in the atoms in the core become close enough together to generate **Fermi pressure** – this occurs because of the Pauli exclusion principal, which states that two electrons cannot be in the same place at the same time.
- A very small, hot and dense star is produced – a **white dwarf**.
- The white dwarf cools gradually, becoming dimmer.

(b) Mass of core between 1.4 and 3 solar masses
(main sequence mass between 5 and 15 solar masses)

- Further thermonuclear reactions are ignited in the compressed core. The first of these is **carbon burning,** which produces neon and magnesium nuclei. This may be followed by further reactions, if the initial mass of the star, and hence temperature reached, is sufficient. These are **neon burning, oxygen burning** and **silicon burning**. The product of silicon burning is iron; this is very stable and so energy would be required for further fusion reactions to take place.
- When the fuel for the final thermonuclear reaction is exhausted, the star will undergo gravitational collapse.
- For stars of this mass, the Fermi pressure of the electrons is not sufficient to prevent further contraction. The electrons combine with protons in the nuclei of atoms to produce neutrons.
- The collapse then continues until the Fermi pressure of the **neutrons** prevents further contraction. This final collapse is extremely rapid and produces a fast rise in temperature.
- The halting of the final collapse produces a shock wave, which with the pressure from the hot core of the star causes it to explode, producing a **supernova**.
- The supernova emits enormous quantities of radiation (sometimes as much as an entire galaxy!) for a few days. The temperatures inside it cause thermonuclear reactions that require energy, rather than give it out. This is how elements heavier than iron are formed.
- The remains of the supernova form a **nebula**.

In some cases, some of a star's core will remain after a supernova. This is a **neutron star** – a star consisting of neutrons as tightly packed as the Fermi pressure will allow. Neutron stars are super-dense – over 10^8 times as dense as a white dwarf, or 10^{14} times as dense as the Earth.

(c) Mass of core greater than 3 solar masses
(main sequence mass greater than 15 solar masses)

- As in (b), the star undergoes gravitational collapse, producing a neutron star.
- For stars of this mass, the Fermi pressure of the neutrons is not sufficient to prevent gravitational collapse. The star collapses further to form a **black hole**.
- The core of the star shrinks to an infinitesimally small point with an infinitely high density – a **singularity.**
- The gravitational field close to the singularity is so strong that even light cannot escape. The boundary of this region – where the escape velocity is equal to the speed of light – is called the **event horizon**.

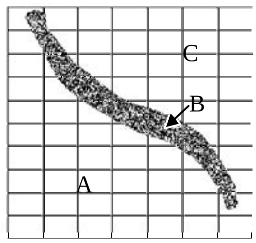
How does the size of a star affect its lifespan?

The lifespan of a star on the main sequence is determined by its mass – the more massive the star, the shorter its lifetime.

Massive stars have a short lifetime – although they have more nuclear fuel available, their luminosity is much greater, and so the fuel is exhausted more quickly.

Typical Exam Question

The diagram below shows an outline Hertzsprung-Russell diagram.



(a) Label the axes of this diagram

(b) Points A, B and C on the diagram are approximate positions of the sun in three stages of its life.

(i) Name the regions of the H-R diagram that A, B and C lie in

(ii) State the sequence in which the sun is expected to occupy these positions.

Solution

(a) x-axis: temperature (or spectral type)
y-axis: absolute magnitude (or intensity)

(b) (i) A: white dwarfs B: main sequence C: red giants
(ii) B C A

How big is a black hole?

The size of a black hole – its **Schwarzschild radius** – can be found by considering the escape velocity.

If a mass m is to just escape the gravitational field of a star of mass M, then it will have zero speed "at infinity". Since gravitational potential energy is also zero at infinity, the total energy of the mass is zero.

Considering its energy at the surface, gives $\frac{mv_{esc}^2}{2} - \frac{GMm}{r} = 0$

For a black hole, $v_{esc} = c$ and $r = R_{sch}$. Rearranging: $R_{sch} = \frac{2GM}{c^2}$

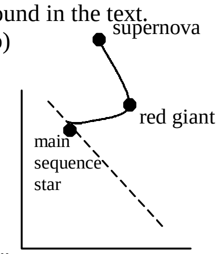
Questions

1. Explain why a blue-white star is hotter than a red star.
2. Sketch an H-R diagram, labelling main sequence stars, red giants and white dwarfs. State where on your diagram the most massive main sequence stars are to be found.
3. Describe the formation of a star up to the point where it joins the main sequence.
4. A blue star has mass 20 times that of the sun.
   (a) State how its lifespan on the main sequence will compare with that of the sun.
   (b) Sketch an H-R diagram to show the life history of this star up to the point where it becomes a supernova.
   (c) Explain what will happen to the core remnant of this star.
5. Calculate the length of a light year in metres, showing your working and giving your answer correct to 1 significant figure.
   (speed of light $c = 3.00 \times 10^8$ ms^{-1})
6. Find the Schwarzschild radius for a black hole of mass 4×10^{30}kg
   (gravitational constant $G = 6.67 \times 10^{-8}$ N m^{2}kg^{-2})

Answers

Answers to questions 1 – 3 may be found in the text.

4. (a) Shorter than the sun (b)
   (b)



   (c) See text for "Death of the star (c)"
5. Number of seconds in 1 year = $60 \times 60 \times 24 \times 365$
   So one light year = $60 \times 60 \times 24 \times 365 \times 3 \times 10^8 = 9 \times 10^{15}$ m
6. $R_{sch} = \frac{2GM}{c^2} = = \frac{2 \times 6.67 \times 10^{-8} \times 4 \times 10^{30}}{9 \times 10^{16}} = 5.9 \times 10^6$m

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Physics Factsheet

April 2003 Number 51

Electromagnetic Doppler effect and the expanding Universe

This Factsheet will explain:
- The theory of the Doppler effect as it affects electromagnetic waves;
- Emission and absorption spectra;
- The red shift of radiation from distant galaxies and Hubble's Law;
- How evidence of the Doppler effect in absorption spectra gives us knowledge of the Universe and its origins.

Before studying this Factsheet, you should be familiar with:
- ideas of the "Red Shift" and the "Big Bang" theory from the GCSE syllabus.

You should also know:
- The relationship: $v = f\lambda$, which connects the velocity, frequency and wavelength of a wave. Factsheet No.17 explores the properties of waves and this relationship.
- That the electromagnetic spectrum is a family of waves, which all travel at the same velocity – the so-called "velocity of light" (3.0×10^8 ms^{-1}), but with different wavelengths; this ranges from waves with the longest (radio waves) with a wavelength of the order of km, to those with the shortest (γ-rays) with a wavelength of the order of pm (10^{-12}m).
- That light (and therefore all the other radiation in the e-m spectrum) is observed to travel at 3.0×10^8 ms^{-1} no matter what the speed of the object which is emitting the radiation.

You should be familiar with principle of superposition of light, the ideas of diffraction and the use of a diffraction grating to produce a spectrum.

You should also be familiar with the idea of the energy levels in an atom, and with the quantization relationship (Factsheet 1):

$\Delta E = hf$ where: ΔE = energy level gap
h = Planck's constant
f = frequency of the emitted radiation

The Doppler Effect

You will have noticed the Doppler effect in sound waves: when a train approaches a station, the pitch (and therefore the frequency) of its whistle rises until it passes an observer on the platform, then falls as the train goes past the observer and moves away. If you hurl a whistle around your head on the end of a piece of string you can hear the changing frequency as the whistle approaches and recedes from your ear.

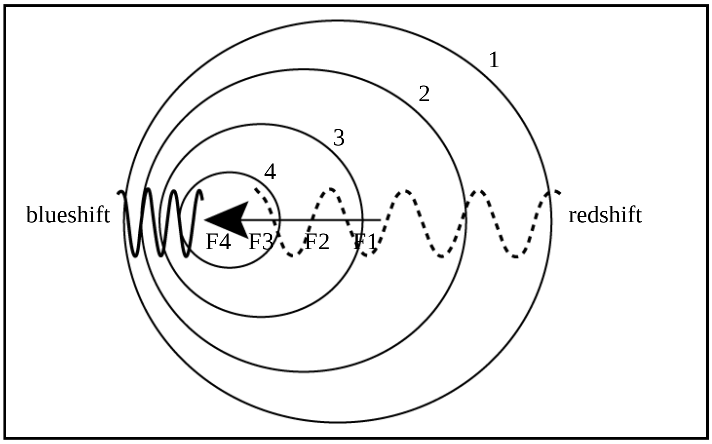
The same effect occurs in electromagnetic waves. The explanation lies in the fact that, unlike a ball thrown from a moving train, the speed of e-m radiation is constant no matter what the speed of the object from which it originates.

Frequency of e-m radiation from a receding object

Consider fig 1. Waves from the object travel with a speed c (3.0×10^8 ms^{-1}), the object is receding with a speed of v, so the waves become spread out over a greater distance, i.e. their wavelength increases. If their wavelength increases while their speed stays the same, then the frequency decreases. The change in wavelength $\Delta\lambda$, causes a change in frequency Δf.

It can be shown that $\dfrac{\Delta\lambda}{\lambda} = \dfrac{\Delta f}{f} = \dfrac{v}{c}$

Fig 1. Frequency of e-m radiation from a receding object

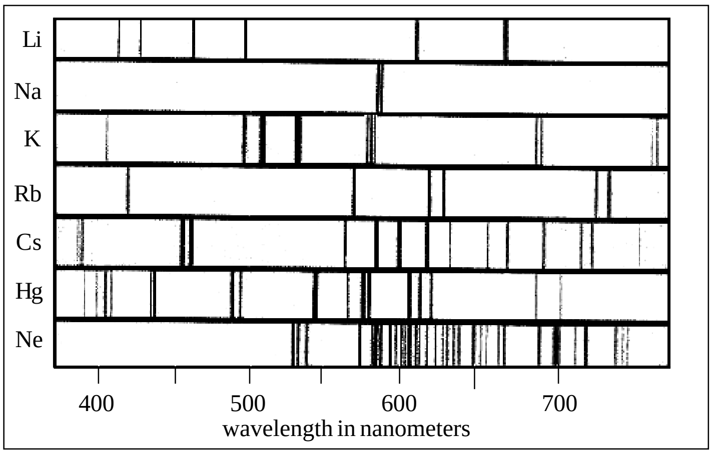


If radiation is emitted from a receding source, its frequency is lowered slightly. The relationship is given by: $\dfrac{\Delta\lambda}{\lambda} = \dfrac{\Delta f}{f} = \dfrac{v}{c}$

Emission Spectra

The spectra of hot gases can be observed using a diffraction grating.

Hot gases emit a spectrum, but it is not a continuous spectrum like a rainbow (called a "black body spectrum"). It consists of discrete lines as shown in the diagram.



When the gas is heated, the electrons are excited into a higher energy state. When they drop back into lower energy states they emit a photon of radiation of a specific frequency. The energy of this photon is given by:

$f = \dfrac{\Delta E}{h}$ where f = the frequency
ΔE = the energy level gap
h = Planck's constant

This gives a line of a frequency characteristic of the element which is emitting the radiation.

An emission spectrum consists of discrete lines of frequency characteristic of the elements emitting the radiation

Example 1
Calculate the frequency of the emission line in a hydrogen spectrum, which corresponds to a transition between energy levels of ΔE = 10.19eV.
Planck's constant $h = 6.6 \times 10^{-34}$Js; $e = 1.6 \times 10^{-19}$C

$$f = \frac{E}{h} = \frac{1.6 \times 10^{-19} \times 10.19}{6.6 \times 10^{-34}} = 2.47 \times 10^{15} Hz$$

Absorption Spectra
When "white light", i.e. radiation of all frequencies, passes through cooler gases, radiation is absorbed at the frequencies where it would have been emitted by the hot gases. So radiation passing through the cooler layers of the atmosphere of distant stars acquires dark lines characteristic of the gases in the atmosphere. Thus we can recognize, say, hydrogen and helium lines in these absorption spectra from distant stars.

An absorption spectrum consists of a "white light" background crossed with dark lines of frequencies characteristic of those elements which have absorbed the radiation.

Doppler shift of lines in absorption spectra.
Studying the absorption spectra of the light from distant stars does indeed show characteristic lines, but the frequencies are shifted by a fixed amount to slightly lower values than those observed close to Earth. (The so-called "Red shift", since the red end of the visible spectrum is of lower frequency than the violet end.) So, for example, if there was a series of hydrogen lines, they would have the same difference in frequencies as those observed with hydrogen spectra from sources on Earth but they would be of slightly lower frequency.

As explained above, the Doppler effect could account for this red shift, if the source of the radiation is moving away from Earth. It is possible to calculate the speed at which the source would need to be moving away in order to give a particular frequency shift, from the formula given above,

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

Doppler effect can explain the shift to lower frequencies, which is observed in the characteristic lines in absorption spectra of the radiation from distant stars. The stars are moving away from each other or receding.

Example 2
Calculate the speed at which a source would be receding, if a line in its spectrum was shifted from a wavelength of 600nm, to 603nm.

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \quad \text{so } v = \frac{3 \times 10^{-9}}{600 \times 10^{-9}} \times 3 \times 10^{8} = 1500 \text{ km s}^{-1}$$

Hubble's Law
The astronomer Edwin Hubble calculated the speeds at which observed galaxies were receding, from the red shift of their absorption spectra . He looked for a relationship between the speeds and the distance away of the galaxy and found that the speed is directly proportional to the distance away. This is expressed as Hubble's Law:

v = Hd *Where: d = the distance of the galaxy from Earth*
v = speed in compatible units
H = the Hubble constant (thought to lie in the range 1.6 - 2.3 × 10^{-18} s^{-1})

Large distances are very difficult to estimate accurately, so there is a large uncertainty in the value of H. (These methods are beyond the scope of A2 study, though methods of estimating distances to the nearest galaxies are dealt with in the Astrophysics Option of some specifications.)

Unit of distance
Since distances in space are so vast they would become unmanageable in m or km. Several other units are used. One is the **light-year**. A light-year is the distance travelled by light in a year.

1 light-year = $3 \times 10^{8} \times 60 \times 60 \times 24 \times 365.25$
*= **9.467×10^{15}m***

The Big Bang
A sensible explanation for the red shift of light from distant galaxies is that they are receding. This supports the Big Bang theory of the origin of the Universe. If the entire Universe began as a concentrated point and then exploded, then we would expect all galaxies to be moving away from each other. Additionally we would expect them to be slowing down, since the only force acting should be the gravitational pull of other galaxies, to reduce the initial velocity after the bang.

Hubble's Law suggests that those galaxies further away from Earth are moving faster. The furthest galaxies are also the oldest, since the light takes longer to reach us, so we are seeing them as they were long ago. Thus the older galaxies are travelling faster.

The red shift of galactic radiation supports the Big Bang Theory

The Age of the Universe
Hubble's Law can also be used to give an approximate value for the age of the Universe. If a galaxy is distance d from our own, and has a recession velocity of v, then separation must have occurred at a time $\frac{d}{v}$ ago.

This represents the approximate age of the Universe.

Since $v = Hd$, $\frac{d}{v} = \frac{1}{H}$

This gives an approximate age of the Universe as 1 – 2 × 10^{10} years (10 – 20 billion years). There are two problems with this:
(a) it assumes a constant speed, and as we have suggested, the speed is thought to have decreased from an initial value to its present value;
(b) the large uncertainty in H.
These factors mean that this is a very approximate estimate!

Typical Exam Question
(a) The table lists three physical quantities. Complete the table.

Physical Quantity	Any commonly used unit	Base units
The Hubble constant		
Galactic distances		
Planck's constant		

[6]

(b) Write a short paragraph explaining how the Hubble constant gives a value for the approximate age of the Universe. [4]

Answer
(a)

Physical Quantity	*Any commonly used unit*	*Base units*
The Hubble constant	$kms^{-1}Mpc^{-1}$ or s^{-1}	s^{-1}
Galactic distances	*light-year*	*m*
Planck's constant	*Js*	kgm^2s^{-1} *

**(energy = force × distance, force is mass × accel, accel is ms^{-2})*

(b) Hubble's Law relates the distance away of distant galaxies with their speed of recession as calculated from the Doppler shift of their radiation, by the equation: v= Hd

Exam Workshop

This is a typical poor student's answer to an exam question. The comments explain what is wrong with the answers and how they can be improved. The examiner's answer is given below.

(a) The Doppler shift may be used to study the movement of distant galaxies. Explain what is meant by a Doppler shift and how it is used to study the movement of distant galaxies. You may be awarded a mark for the clarity of your answer. [5]

Doppler shift is when the wavelength is shifted. It shows that galaxies are moving apart. This supports the Big Bang theory. 1/5

> The student has been award a mark for mentioning movement away associated with a shift of wavelength, but has failed to identify what wavelength s/he is talking about, or which way it is shifted. No explanation of the Doppler effect has been given. No credit has been given for the reference to the Big Bang theory, since this is not called for in the question.

(b) Hubble's law states that: $v = Hd$
where d is the distance of a galaxy from Earth, and v its speed in compatible units, and H is the Hubble constant.

(i) Give a suitable unit for H [1]

v is in light-years and d in m, so H is light-years/m 0/1

> The student has become confused. The light-year is a unit of distance, not time or speed.

(ii) Explain why there is a large uncertainty in H [2]

because the quantities are difficult to measure accurately 1/2

> While the student appreciates that difficulties in measuring cause an uncertainty, s/he has not been sufficiently specific that it is the DISTANCE which is difficult to measure with accuracy.

(iii) Explain how the Doppler shift and Hubble's Law support the Big Bang theory. [2]

Galaxies are moving away from each other, so must have exploded in the past. 1/2

> Not sufficient for both marks.

Examiner's Answers

(a) When EM radiation is emitted from a moving source, ✓ the discrete spectral lines in its absorption spectrum suffer a change ✓ of wavelength.

If the source is moving away, then the shift is to longer wavelengths. ✓
The speed of recession can be calculated from $\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$ ✓ ✓

(b) (i) If d is in km, then compatible units for v would be km s^{-1}, so units of H are s^{-1}. ✓

(ii) To determine H requires measurement of $\lambda, \Delta\lambda$, ✓ and d.
Measurement of λ and $\Delta\lambda$ can be done with reasonable accuracy, but d ✓ is very difficult to measure accurately.

(iii) The galaxies that are furthest away are also the oldest, since light has taken longest to reach us. ✓
These are moving faster, so younger, closer galaxies are moving at slower speeds, as would be expected if they had been produced in a huge ✓ explosion at some stage in the past.

Questions

1. Explain why radiation from a moving source experiences a shift of wavelength.
2. Give the equation for the shift of wavelength.
3. Is light from a receding source shifted to longer or shorter wavelengths?
4. State Hubble's Law.
5. A spectral line in the absorption spectrum of a distant galaxy shows a Doppler shift from 550nm to 555nm.
   (a) Calculate the speed of recession of the galaxy.
   (b) If the galaxy's distance away is estimated to be 1.44×10^{21}km, what value does this give for the Hubble constant?

Answers

1. See text
2. See text
3. Longer
4. $v = Hd$
5. a) $\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$ so $v = \frac{5 \times 3 \times 10^8}{550} = 2.7 \times 10^6$ m/s

b) $v = Hd$, so $H = \frac{v}{d} = \frac{2.7 \times 10^6}{1.44 \times 10^{24}} = 1.9 \times 10^{-18}\ s^{-1}$

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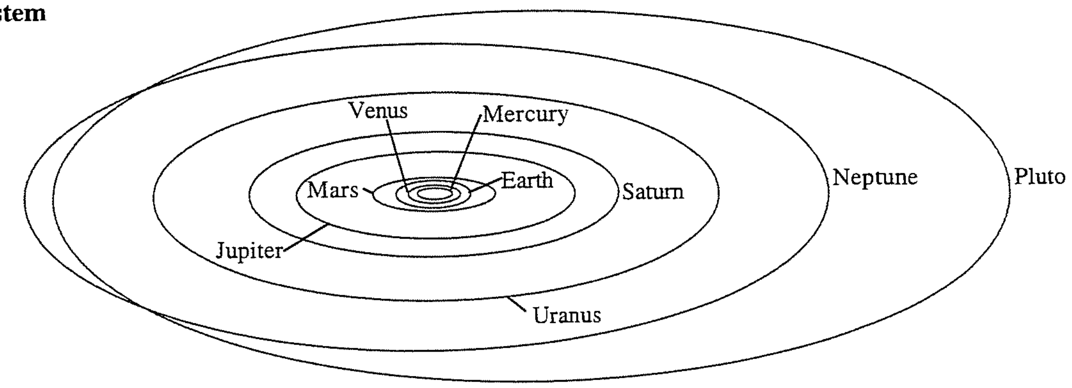
An Inventory of the Solar System: Part I

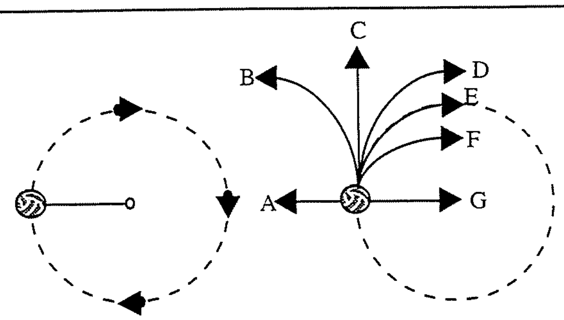
In this factsheet we will explore circular motion, centripetal acceleration and centripetal force in everyday situations, and apply these ideas to our solar system.

Our solar system consists of one sun, nine planets, over 150 natural satellites and many small bodies like asteroids and comets. The four inner planets are relatively small, rocky bodies. The four outer planets are gas giants. Pluto can be considered a bit of an anomaly. All of the nine planets have elliptical orbits: elongated circles.

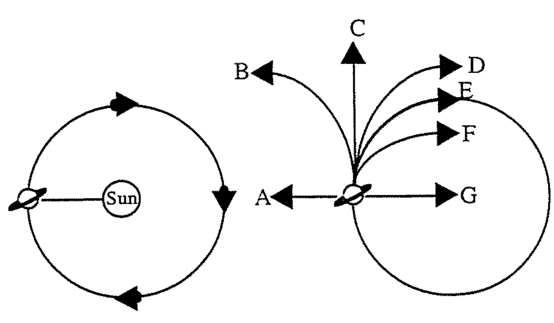
The Solar System



Spin a ball at the end of a piece of string around your head. Someone cuts the string. What happens next? What would happen to a planet if the sun disappeared?



If you swing a ball on a string around your head, when the string is cut where does the ball go?



A planet orbits the sun, if the sun is removed where does the planet go?

Both the ball **AND** the planet would travel along path C. Why? If we cut the string or removed the sun, there would be **no force** acting on the ball or planet. We know what happens if there is no resultant force acting: the ball or planet stays still or carries on with the same steady speed in a straight line (Newton's 1st Law of motion).

Without the string, the ball would follow path C. With the string, the ball follows path E. There **must** be a force acting on the ball, constantly pulling the ball around into a circular path. What is this force? Which direction does it act?

The force is inwards, towards the centre of the circle. It is caused by the tension in the string.

> *Any object travelling in a circle will continue in a straight line at the same steady speed if the inward force is removed.*

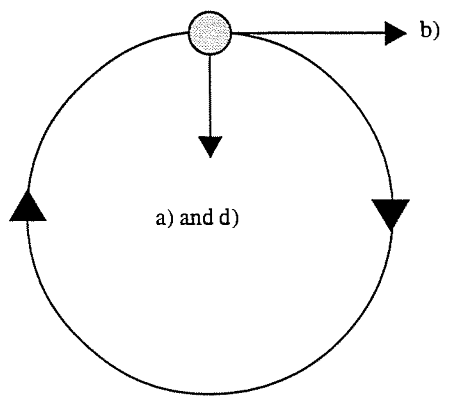
We know acceleration happens when an object changes velocity. But velocity is a vector: speed in a particular direction. If the direction of travel changes, we have acceleration, even if the actual speed remains the same. So an object moving in a circle is ***always*** accelerating towards the centre, even if the speed stays the same.

This is also true for any object in an orbit. The gravitational attraction between the sun and the planet keeps the planet in an orbit. The planet is continually accelerating, but travels at a steady speed! When something is travelling in a circle, we call the inward acceleration **centripetal acceleration** and the inward force **centripetal force**. These aren't NEW forces, just a handy way of thinking about the forces already involved, like tension in a string, friction on a road or gravity between a sun and planet.

An object moving in a circle is always being accelerated, but the speed DOES NOT change

Example Exam Question
A yo-yo is spun at a steady speed *around a person's head* at the end of a piece of string.
(a) Draw an arrow on the diagram showing where the force is acting. (1 mark)
(b) Draw an arrow to show where the yo-yo would travel if the string were cut. (1 mark)
(c) Explain why the yo-yo is accelerating, despite travelling with a steady speed. (2 marks)
(d) Draw an arrow to indicate the direction of acceleration. (1 mark)

Answer



a) ✓ b) ✓ and d) ✓
c) Velocity is speed in a particular direction. ✓ But the direction keeps changing so the yo-yo must be accelerating. ✓

Centripetal acceleration and centripetal force

Centripetal acceleration $a = \frac{v^2}{r}$

If the radius is small, we need a big acceleration. We need a very big acceleration if the speed is great because of the v^2 term. The planets closest to the Sun move fastest. A large centripetal acceleration is required to keep them in orbit.
This is the relationship for centripetal acceleration, speed and radius.

Newton's 2nd law of motion tells us F=ma. This gives us a relationship for centripetal force:

Centripetal force $F = \frac{mv^2}{r}$

So for the fast moving inner planets, a large centripetal force is required. The Sun's gravitational field is strongest nearest the inner planets, and is the source of this centripetal force.

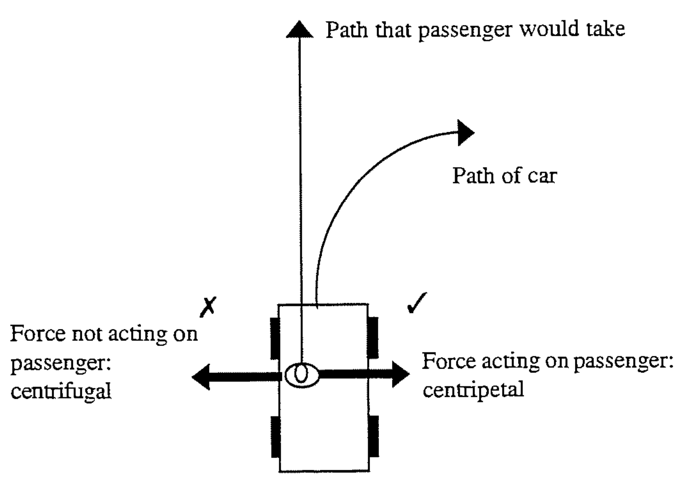
For any object moving a circle, the force is ALWAYS towards the centre

Exam Hint: *Q) Does the earth do work keeping the moon in its orbit? A) The force is always acting at right angles to the motion. No work is done, because there is no component of the force in the direction of motion.*

Exam Hint: *What happens if you twirl a yo-yo around your head at the end of its string and let go? It will continue in a straight line in the direction it was travelling when released. But don't forget it will ALSO fall down towards the ground: accelerating due to gravity. Its path will be parabolic in a vertical plane.*

Misconception: Centripetal and centrifugal

- Picture yourself sitting in the passenger seat of a car. What happens to you when the car turns right?
- You feel pressed up against the door. Many people say there is a force "throwing you" outwards.
- However, there is NO OUTWARD FORCE
- Your body is "trying" to obey Newton's First Law of motion: continue at the same speed in the same direction. The force acts ***inwards***: forcing your body around the corner



Practice Questions

Show all working, including any rearranging of equations.

1. What is the cause of the centripetal force keeping planets in their orbits? Which direction does it act?
2. Describe and explain what would happen to any object moving in a circular path if the centripetal force were removed.
3. The radius of the moon's orbit around the earth is 384,000km. Calculate the distance it travels in one orbit. The orbital period of the moon is 27.3 days. How fast does the moon travel, in metres per second?
4. Calculate the centripetal acceleration of the moon using your answer from question 3.
5. The earth has a mass of 5.98×10^{24}kg, an orbital speed of 30kms^{-1} and orbits 150 million km from the sun. Explain why we can use the centripetal force equation to calculate the gravitational attraction of the earth to the sun. Calculate this attraction.
6. How can an object be accelerating if it moves at a steady speed in a circular path?
7. If the speed of a car is doubled, how does the centripetal force needed for it to turn a corner change? What happens to required centripetal force if the radius of the turn is halved? What if the mass of the car is halved?
8. If a circular fairground ride is limited to 4g for safety reasons, with what linear speed can it spin? The radius of the ride is 10m and g =9.8ms^{-2}.

Answers

1) Gravity provides the centripetal force keeping the planets in their orbits. It acts inwards, towards the centre of the circular path.

2) An object will travel in a straight line in the direction it was travelling at the instant the centripetal force is removed.

3) Circumference of moons orbit = $2\pi r$ where $r = 3.84 \times 10^8$m so circumference = 2.41×10^9m. Period of the moons orbit =27.3 days $= 27.3 \times 24 \times 60 \times 60 = 2.36 \times 10^6$ seconds.

   Moons orbital speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{2.41 \times 10^9\text{m}}{2.36 \times 10^6\text{s}} = 1020\text{m/s}$

4) Centripetal acceleration $= \dfrac{v^2}{r} = \dfrac{1020^2}{3.84 \times 10^8} = 2.71 \times 10^{-3}\text{ms}^{-2}$

5) The centripetal force keeping the earth in its orbit is caused by the gravitational attraction between the earth and the sun.

   $F = \dfrac{mv^2}{r} = \dfrac{5.98 \times 10^{24}\text{kg} \times (30 \times 10^3\text{ m/s})^2}{150 \times 10^9\text{m}} = 3.59 \times 10^{22}\text{N}$

6) Acceleration is due to a change in velocity. Velocity is speed in a particular direction. A circular path means the direction is constantly changing and so is the velocity.

7) Centripetal force is given by $F = \dfrac{mv^2}{r}$

   If the speed of the car is doubled, the force needed for the turn is quadrupled because $F \propto v^2$. If the radius of the turn is halved, the force required for the turn is doubled because $F \propto \dfrac{1}{r}$

   If the mass of the car is halved, the force required for the turn is halved because $F \propto m$.

8) If g = 9.8ms^{-2} then maximum centripetal acceleration = 4g = 39.2ms^{-2}.

   $a = v^2/r$ so $v = \sqrt{(a \times r)} = \sqrt{(39.2 \times 10)} = 19.8\text{ms}^{-1}$

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An Inventory of the Solar System: Part II

Gravity
Gravity is an attractive force between masses. The attraction is bigger if the masses are large, or if they are close together.

Newton's Law of Gravitation $F = \frac{-Gm_1m_2}{r^2}$

But the gravitational constant, G is very small (6.67×10^{-11} Nm^2kg^{-2}). The gravitational attraction between two people is tiny.

Gravitational field strength
Any object with mass has a gravitational field around it. Anything with mass experiences a force in a gravitational field. We need some way of comparing the gravitational fields of, say, the earth and the moon. We use gravitational field strength, g.

Gravitational field strength $g = \frac{F}{m}$

Gravitational field strength is the gravitational force exerted on a unit mass. Everything on the surface of the earth experiences a gravitational field strength of 9.8 Newtons per kilogram. Everything on the surface of the moon experiences a gravitational field strength of 1.6 Newtons per kilogram.

We can substitute for F/m in Newton's law of gravitation to give us another equation for gravitational field strength.

Gravitational field strength $g = \frac{-GM}{r^2}$
Where M is a spherical mass such as a star or planet.

***Exam Hint:** Students often use the word "gravity" on its own. Quite often this does not gain credit. Make it clear whether you are talking about gravitational field strength, gravitational force, gravitational potential, or gravitational potential energy.*

When we think of how fast a planet moves around in its orbit, the planetary mass is irrelevant. ANY planet 150 million km from the sun would experience the same gravitational force per kg as earth.

The inner planets move more rapidly because the sun's gravitational field strength is greater.

How does orbital speed change with distance from the sun? Why?

$F=GMm/r^2$ So $F\propto 1/r^2$

If the radius of orbit is doubled, then the gravitational force acting on an object is reduced four-fold.

Gravity acts as the centripetal force for an object moving in an orbit, so we can also use $F=mv^2/r$.

$mv^2/r = GMm/r^2$

Rearranging this gives $v^2 = GM/r$ or $v = \sqrt{(GM/r)}$

So if the orbital radius is doubled, the speed of the orbit is reduced by a factor of $\sqrt{2}$.

For example, Saturn orbits 3.1 times more slowly than Earth and 9.5 times further from the sun. If the orbital radius is 9.5 times greater, the speed is $\sqrt{9.5}$ times slower, $\sqrt{9.5} = 3.1$!

***Exam Hint:** Often when data is presented in a table, much of the data is not needed for the questions asked. Don't assume that you have to make use of every fact presented to you. Decide what you need to use, and ignore the rest.*

Table: Data on our solar system

	Orbital Radius (AU)	Mass (earths)	Diameter (earths)	Orbital Period (years)	Surface Gravity (earths)	Orbital speed (km/s)
Sun	0.0	330,000	109.2	-	28	
Mercury	0.4	0.06	0.38	0.24	0.38	**49.8**
Venus	0.7	0.81	0.95	0.62	0.90	**33.7**
Earth	1.0	1.00	1.00	1.0	1.00	**29.9**
Mars	1.5	0.11	0.53	1.9	0.38	**23.6**
Jupiter	5.2	317.8	11.2	11.9	2.34	**13.1**
Saturn	9.5	95.2	9.4	29.4	1.16	**9.7**
Uranus	19.2	14.5	4.0	83.7	1.15	**6.9**
Neptune	30.1	17.2	3.9	163.7	1.19	**5.5**
Pluto	39.4	0.002	0.18	248.0	0.04	**4.7**

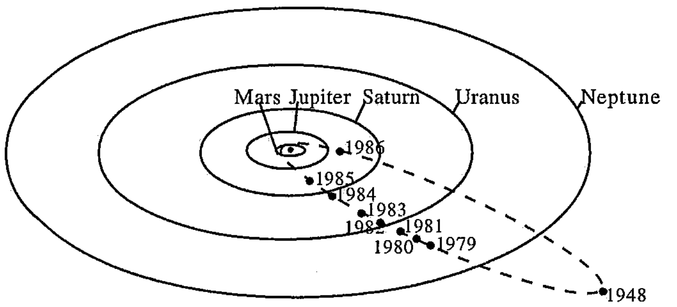
Ellipses and comets

All of the planets have elliptical orbits: elongated circles. Only Pluto has an orbit which is noticeably elliptical: sometimes it is closer to the sun than Neptune!

Occasionally a comet can fall towards the inner solar system. As it nears the sun, some of the particles and gases are thrown out in a spectacular tail. The orbit is very elliptical.

Halley's Comet

The orbit of Halley's Comet is roughly 76 years. Notice the speed of the comet as it approaches the sun. (Check the distance travelled between 1985 and 1986.)



Why does it accelerate so much in the inner solar system? (The gravitational field strength is proportional to $1/r^2$.)

When the comet is nearer to the sun, it experiences a much greater force per unit mass: big acceleration on the way in and big deceleration on the way out. In the outer solar system, the gravitational field strength is much less, causing small changes in speed. It was moving slowest in 1948 and fastest in 1986.

As every planet has a slightly elliptical orbit, it moves slightly faster when nearer the sun and slightly slower when further from the sun.

Example exam question
Describe how the speed of a comet varies throughout its path. Explain this variation, referring to the gravitational field strength of the sun.

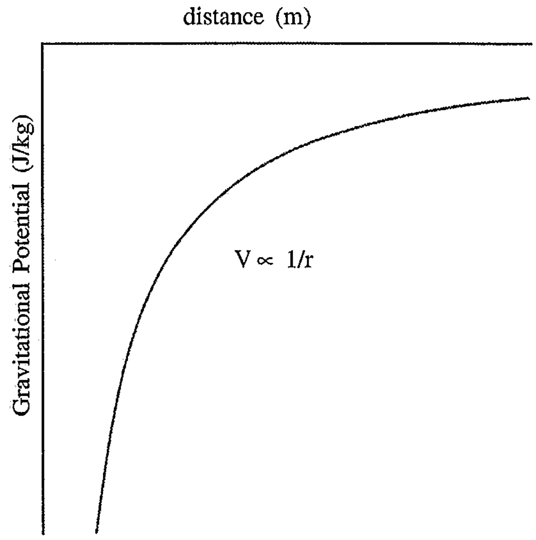
Example Answer
A comet has an elliptical orbit.
It moves quickly in the inner solar system and slowly in the outer solar system. ✓
Gravitational field strength is proportional to $1/r^2$. g is high in the inner solar system, causing large changes in speed. g is weak in the outer solar system, causing smaller changes in speed. ✓

Exam Hint: *DON'T ALWAYS use mgΔh to calculate an increase in gravitational potential energy. Remember that gravitational field strength falls over big distances. Use V=-GM/r when the change in height is hundreds or thousands of kilometres.*

Gravitational Potential

If we lift an apple off the ground, we increase its Gravitational Potential Energy (GPE): $\Delta E_p = mg\Delta h$. We are doing work, applying a force over a distance. But as we lift the apple higher and higher, earth's gravitational field is becoming weaker and weaker: $g \propto 1/r^2$.

If we could calculate the amount of work done lifting an apple off the earth, we could find the GPE. Work done = force × distance. The potential energy of an apple is low on earth's surface. It increases as you move away. Finally, the GPE of the apple increases to zero if you move it far enough away. What would a graph of this look like?



Force acting **on a unit mass** $F = GM/r^2$
Work done = force × distance
Work done moving **this unit mass** to infinity is the gravitational potential.

Gravitational potential (J/kg) $V = \frac{-GM}{r}$

Exam Hint: *Make sure any sketch of gravitational force against distance ($F \propto 1/r^2$) drops more rapidly than a sketch of gravitational potential against distance ($V \propto -1/r$) rises*

Gravitational field strength units are N/kg. Gravitational potential units are J/kg.

Example exam question
Calculate the gravitational potential energy of a cricket ball on the earth' surface. State how much energy is needed to move the ball to infinity.
Earth mass = 5.98×10^{24}kg Earth radius = 6370km Cricket ball = 0.5kg

Example Answer
$V = -GM/r = -6.67 \times 10^{-11} \times 5.98 \times 10^{24}/6370 \times 10^3 = -6.23 \times 10^7$ J/kg ✓
Gravitational potential energy of ball = J/kg × 0.5kg = -3.1×10^7J ✓
Energy to move to infinity = 3.1×10^7J ✓

Escape velocity

How fast would you need to throw a ball so it never came back down? All of the initial kinetic energy would eventually become GPE. If enough KE is provided, the ball could reach infinity.

KE = GPE $1/2mv^2 = GMm/r$
or $1/2v^2 = GM/r$
or $v = \sqrt{(2GM/r)}$ (escape velocity)

Example exam question
Calculate the escape velocity for earth.
The mass of the earth = 5.98×10^{24} kg.
The average radius of the earth = 6370km.

Example Answer
$1/2mv^2 = GMm/r$ ✓ $1/2v^2=GM/r$ $v= \sqrt{(2GM/r)}$ ✓

$$v = \sqrt{\frac{(2\times6.67\times10^{-11} \times 5.98\times10^{24})}{6370\times10^3}} = 11.2km/s \checkmark$$

Exam Hint: *It is a common misconception that an object needs to reach escape velocity to leave the atmosphere. This is only true if there is no continuous push. A rocket could travel out of the atmosphere very slowly, if its engines were constantly pushing it forward. Don't assume rockets have to reach high speeds.*

Artificial satellites

Different artificial satellites have different purposes. We may need a satellite to always appear in one part of the sky to provide satellite TV. We may need a satellite that orbits earth rapidly: as the earth spins below it, the whole of the globe can be imaged or studied. It is the mass of the planet and the height of the satellite which determines the speed of the satellite:

Radius and period of any orbit around a mass, M

$$r^3 = \frac{GMT^2}{4\pi^2}$$

We can calculate the necessary altitude of any satellite travelling in a circular path around earth if we know the orbital period. For an orbital period of 90 minutes:

$r^3 = 6.67\times10^{-11} \times 5.98\times10^{24}\times(90*60)^2/4?^2$, $r = 6.654\times10^6$m or 6654km
This is 6654 – 6370 = 284km altitude

Example exam question
Calculate the orbital altitude of a geostationary satellite.

Example Answer
$F=GMm/r^2 = mv^2/r$ $v^2=GM/r$ ✓✓
Time for 1 orbit = $2\pi r/v$ so $v = 2\pi r/T$ ✓

$$r^3 = \frac{GMT^2}{4\pi^2}, \quad T = 24 \times 60 \times 60 \text{ seconds} \checkmark$$

$$r^3 = 6.67\times10^{-11} \times 5.98\times10^{24} \times \frac{(24 \times 60 \times 60)^2}{4\pi^2} = 7.54\times10^{13}$$

r = 42249 km or an altitude of 35880km ✓
(This has been worked out without assuming the equation for r^3.)

Questions

Show your working clearly for every question. Think carefully about units before calculations. Mass of the earth = 5.98×10^{24}kg. Mass of the moon = 7.35×10^{22}kg. Earth moon separation = 3.8×10^5km. Earth's radius = 6370km (approximately). All other information can be calculated from the table on page 1.

1. Use Newton's Law of Gravitation to calculate the gravitational attraction between the earth and the moon.
2. Calculate the gravitational attraction between the sun and the earth.
3. Use the equation for gravitational field strength to calculate the surface gravity for Earth, Mars and Jupiter. Compare your answers to the surface gravity data in the table. Remember to use planetary radius, not diameter.
4. Describe and explain the change in the orbital velocity of the planets throughout the solar system.
5. Describe and explain the speed of a comet throughout its orbit.
6. Calculate the gravitational potential of a geostationary satellite. The radius of a geostationary orbit is 42250km. What is the change in potential from lift-off?
7. Calculate the escape velocity for Mercury, Earth and Jupiter.

Answers

1. $F = \frac{-Gm_1m_2}{r^2} = 6.67\times10^{-11} \times 5.98\times10^{24} \times \frac{7.35\times10^{22}}{(3.8\times10^8)^2} = 2.03\times10^{20}N$

2. $F = \frac{-Gm_1m_2}{r^2} = 6.67\times10^{-11} \times 5.98\times10^{24} \times \frac{1.97\times10^{30}}{(1.5\times10^{11})^2} = 3.49\times10^{22}N$

3. Earth, 9.8N/kg. Mars, 3.9N/kg. Jupiter, 24.9N/kg

4. The force of gravitational attraction is proportional to $1/r^2$. If the distance from the sun is doubled, the gravitational attraction is reduced by a factor of 4. Orbital speed is proportional to $1/\sqrt{r}$, so if the radius is doubled, the speed is reduced by $1/\sqrt{r}$.

5. A comet has a very elliptical orbit. In the inner solar system, the gravitational field strength of the sun is great, causing large changes in speed: accelerating inwards then decelerating outwards. In the outer solar system, the suns gravitational field strength is much weaker, causing smaller changes in speed.

6. $V = \frac{-GM}{r} = -6.67\times10^{-11} \times \frac{5.98\times10^{24}}{4.22\times10^7} = -9.45\times10^6 J/kg$

   $V = \frac{-GM}{r} = -6.67\times10^{-11} \times \frac{5.98\times10^{24}}{6.37\times10^6} = -6.26\times10^7 J/kg$

   Change in potential = 5.32×10^7 J/kg

7. Escape velocity is given by $v = \sqrt{(2GM/r)}$.
   For Mercury, $v = \sqrt{\frac{2 \times 6.67\times10^{-11} \times 3.59\times10^{23}}{2.42\times10^6}} = 4.4$ km/s
   Earth, 11.2km/s. Jupiter 59.6km/s.

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